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An analog of the Skewes number for twin primes

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Abstract

The results of the computer investigation of the sign changes of the difference between the number of twin primes $\pi_2(x)$ and the Hardy–Littlewood $c_2\mathrm{Li}_2(x)$ conjecture are reported. It turns out that this difference changes the sign at unexpectedly low values of x and for $x < 2^{42}$ there are over 90000 sign changes of the $\pi_2(x) - c_2\mathrm{Li}_2(x)$. It is conjectured that the number of sign changes of $\pi_2(x) - c_2\mathrm{Li}_2(x)$ for $x \in (1, T)$ is given by $\sqrt{T}/\ln(T)$.

Let $\pi(x)$ be the number of primes less than x and let $\mathrm{Li}(x)$ denote the logarithmic integral:

$$\mathrm{Li}(x) = \int_2^x \frac{du}{\ln(u)}. \tag{1}$$

The Prime Number Theorem tells us that $\mathrm{Li}(x)/\pi(x)$ tends to 1 for $x \to \infty$ and the available data (see [2], Table I) shows that always $\mathrm{Li}(x) > \pi(x)$. This last experimental observation was the reason of the common belief in the past, that the inequality $\mathrm{Li}(x) > \pi(x)$ is generally valid. However, in 1914 J.E. Littlewood has shown [1] that the difference between the number of primes smaller than x and the logarithmic integral up to x infinitely often changes the sign. The problem of the sign changes was treated subsequently by A.E. Ingham [3], chapter V, [4] and S. Knapowski [5]. They were particularly interested in the number of the sign changes of the difference $d(x) = \pi(x) - \mathrm{Li}(x)$ for x in a given interval $(1, T)$, which is commonly denoted by $\nu(T)$, see [6]. Regarding the number of sign changes of $d(x)$ in the interval $(1, T)$, Knapowski [5] proved

$$\nu(T) \geq e^{-35} \ln\ln\ln\ln T \tag{2}$$

provided $T \geq \exp\exp\exp\exp(35)$.

In 1933 S. Skewes assuming the truth of the Riemann hypothesis has estimated that for sure $d(x)$ changes sign for some $x_0 < 10^{10^{10^{34}}}$. The smallest value x_0 such that for the first time $\pi(x_0) > \mathrm{Li}(x_0)$ holds is called Skewes number. In 1955 Skewes [8] has found, without assuming the Riemann hypotheses, that $d(x)$ changes sign at some

$$x_0 < \exp\exp\exp\exp(7.705) < 10^{10^{10^{10^3}}}.$$

This enormous bound for x_0 was reduced by Cohen and Mayhew [9] to $x_0 < 10^{10^{529.7}}$ wihout using the Riemann hypothesis. In 1966 Lehman [10] has shown that between 1.53×10^{1165} and 1.65×10^{1165} there are more than 10^{500} successive integers x for which $\pi(x) > \mathrm{Li}(x)$. Following the method of Lehman in 1987 H.J.J. te Riele [11] has shown that between 6.62×10^{370} and 6.69×10^{370} there are more than 10^{180} successive integers x for which $d(x) > 0$. In this paper I am going to look for the analog of the Skewes number for the twin primes.

Let $\pi_2(x)$ denote the number of twin primes smaller than x. Then the unproved (see however [13]) conjecture B of Hardy and Littlewood [12] on the number of prime pairs $p, p+d$ applied to the case $d = 2$ gives, that

$$\pi_2(x) \sim c_2 \mathrm{Li}_2(x) \equiv c_2 \int_a^x \frac{u}{\ln^2(u)} du, \tag{3}$$

where c_2 is called "twin constant" and is defined by the following infinite product:

$$c_2 \equiv 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) = 1.32032\ldots \tag{4}$$

Usually the lower limit of integration a in (3) is chosen 2, but the author believes that the proper choice for the lower limit of integration should be 5, not 2, because (3,5) is the first twin pair. (Analogously in (1) the lower limit of integration is 2, to ensure that $\mathrm{Li}(2) = 0$.)

TABLE I

The number of sign changes of $d_2(x)$

The case $a = 2$ is in second column, and third column contains data for $a = 5$.

T	$\nu_2(T)$ for $a = 2$	$\nu_2(T)$ for $a = 5$	$\frac{\sqrt{T}}{\ln(T)}$
2^{22}	29	32	134
2^{23}	29	32	182
2^{24}	29	32	246
2^{25}	29	32	334
2^{26}	238	269	455
2^{27}	854	942	619
2^{28}	1226	1401	844
2^{29}	1226	1401	1153
2^{30}	1226	1401	1576
2^{31}	1226	1401	2157
2^{32}	2854	3045	2955
2^{33}	7383	7358	4052
2^{34}	9115	8974	5562
2^{35}	12682	12431	7641
2^{36}	23634	23103	10505
2^{37}	31641	30770	14455
2^{38}	31641	30770	19905
2^{39}	31641	30770	27428
2^{40}	38899	37904	37819
2^{41}	55106	54179	52180
2^{42}	90355	n.a.	72037

I have looked on the difference $d_2(x) = \pi_2(x) - c_2\mathrm{Li}_2(x)$ using the computer for T up to $2^{42} \approx 44 \times 10^{12}$. Surprisingly, it turns out that there is a lot sign changes of $d_2(x) = \pi_2(x) - c_2\mathrm{Li}_2(x)$ for x in the range $(1, 2^{42})$. For the case of the lower integration limit $a = 2$ in (3) the first sign change of $d_2(x)$ appears at the twin pair (1369391, 1369393). However, for the choice $a = 5$ the first sign change of $d_2(x)$ appears already at the pair (41, 43)! Next "Skewes" pairs are (6959, 6961) and (7127, 7129). After that there is a gap in the sign changes up to (1353257, 1353259), what is comparable to the case of $a = 2$. Such a behavior is reasonably, because the difference between $a = 2$ and $a = 5$ is relevant only at low x and the contribution stemming from the interval (2,5) is becoming negligible for large x.

Let $\nu_2(T)$ denote, by analogy with usual primes, the number of sign changes of $\pi_2(x) - c_2\text{Li}_2(x)$ in the interval $(1, T)$. The Table I contains the recorded number of sign changes of $\pi_2(x) - c_2\text{Li}_2(x)$ up to $T = 2^{22}, 2^{23}, \ldots, 2^{42}$ for both choices of the parameter a. The values of T searched by the direct checking are of small magnitude from the point of view of mathematics, but large for modern computers. So it is very difficult to fit the function to the numbers in the second and third column in Table I, especially if the looked for formula contains iterated logarithms. The observed numbers $\nu_2(T)$ behave very erratically, see Fig.1, in particular there are large gaps without any change of sign of the $d_2(x)$. Nevertheless the last column contains the values of the function $\sqrt{T}/\ln(T)$, which I have picked up after several trials. If the conjecture

$$\nu_2(T) \sim \sqrt{T}/\ln(T) \tag{5}$$

is true, then there is infinity of twins. Also if (5) is valid, it means that the estimation (3) is in some sense more accurate than (1), because there is more points, where (3) exactly reproduces the actual number of twins — for (1) there is much less such values of x that $\text{Li}(x)$ is equal to $\pi(x)$, see (2). However, presumably the (unknown) error term in (3) is larger than error term for $\pi(x)$. The Fig.1 presents the comparison of (5) with the real data for $a = 2$.

I have tested the numerical results using several different computers, programs and compilers. In particular, to calculate the integral I have used the 8–point self–adaptive Newton–Cotes method and the 10–point Gauss method. It seems to be natural that different methods gave exactly the same values of $\text{Li}_2(x)$ since the integrand in (3) is a very well behaved function. At least up to $T = 2^{32}$ all results obtained by different runs were exactly the same.

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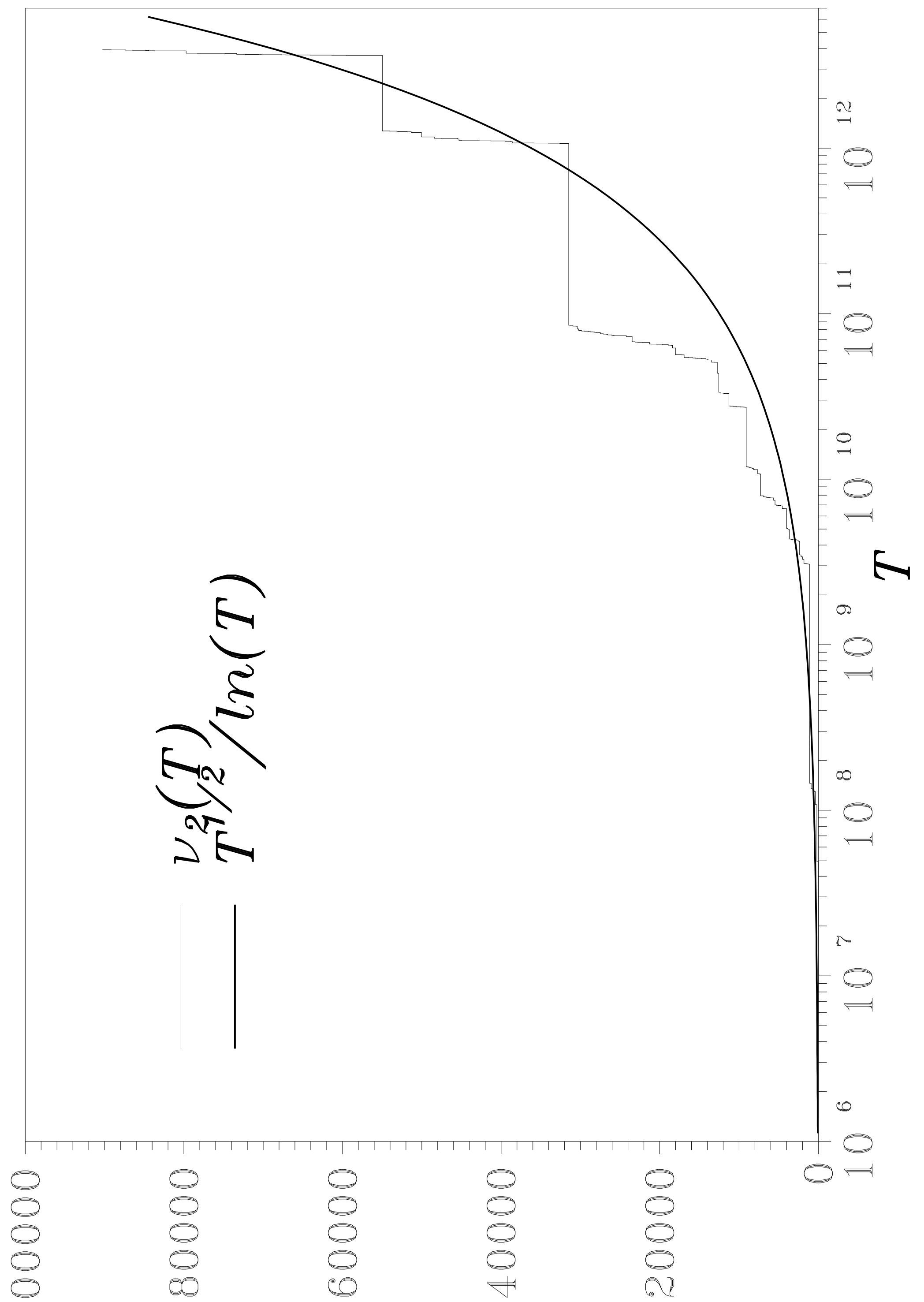

Fig.1 The plot showing the comparison of the actual values of $\nu_2(T)$ found by a computer search (thin line) and the plot of with conjecture (5) drawn by bold line.