April 1997, IFTUWr 911/97

First occurrence of a given gap between consecutive primes

Marek Wolf

Institute of Theoretical Physics, University of Wrocław
Pl.Maxa Borna 9, PL-50-204 Wrocław, Poland
e-mail: mwolf@ift.uni.wroc.pl

Abstract

Heuristic arguments are given, that the pair of consecutive primes separated by a distance d appears for the first time at $p_f(d) \sim \sqrt{d} \exp\left(\frac{1}{2}\sqrt{\ln^2(d) + 4d}\right)$. The comparison with the results of the computer search provides the support for the conjectured formula.

Key words: *Prime numbers, gaps between primes*
PACS numbers: 11A99, 11A41

In my earlier paper [1] it was conjectured, that the sums of reciprocals of all consecutive primes smaller than x and separated by gaps of the length d for $d \geq 6$ are equal to

$$\mathcal{B}_d(x) = \sum_{p \in \mathcal{T}_d(x)} 1/p = \frac{4c_2}{d} \prod_{p|d} \frac{p-1}{p-2} e^{-d/\ln(x)} + \textit{error term}. \tag{1}$$

Here the constant c_2 (sometimes called "twin–prime " constant) is defined in the following way:

$$c_2 \equiv \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) = 0.66016\ldots \tag{2}$$

and $\mathcal{T}_d(x)$ denote the set of consecutive primes smaller than x and separated by distance d:

$$\mathcal{T}_d(x) = \{p_{n+1}, p_n \mid p_{n+1} - p_n = d \quad \text{and} \quad p_{n+1} < x\} \tag{3}$$

where $d = 2k$, $k = 1, 2, \ldots$. I shall call these primes "twins of genus d" and the limits of (1) when $x \to \infty$ the generalized Brun's constants $\mathcal{B}_d$. For $d = 2$ the set $\mathcal{T}_2$ contains twin primes and it was the case studied originally by Brun in 1919 [2], where he has proved that the sum of reciprocals of all twin primes is finite. In [1] it was also shown, that (1) reproduces the leading $\ln(\ln(x))$ term in the sum:

$$\sum_{p<x} \frac{1}{p} = \frac{1}{2} + \frac{1}{6} + \frac{1}{2} \sum_d \mathcal{B}_d(x) = \ln(\ln(x)) + \mathcal{O}(1/\ln(x)) \tag{4}$$

In this note I am going to use the conjecture (1) to estimate the position of the first appearance of a pair of primes separated by gap of the length d. More specifically, let:

$$p_f(d) = \text{minimal prime, such that the next prime } p' = p_f(d) + d. \tag{5}$$

The form of the error term in (1) is unknown, but the comparison with computer data shows that it decreases with d rather fast, see [1]. We can obtain the heuristic formula for $p_f(d)$ by remarking, that the finite approximations to the generalized Brun's constants are for the first time different from zero at $p_f(d)$ and then they are equal to $2/p_f(d)$:

$$\frac{4c_2}{d} \prod_{p|d} \frac{p-1}{p-2} e^{-d/\ln(p_f(d))} = \frac{2}{p_f(d)}. \tag{6}$$

The product $\prod_{p|d} \frac{p-1}{p-2}$ behaves very erratically, but it takes values of the order 1. In fact, its mean value is equal $1/c_2$ [3]:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \prod_{p|2k, p>2} \frac{p-1}{p-2} = \frac{1}{c_2} = \frac{1}{\prod_{p>2}(1 - \frac{1}{(p-1)^2})} \quad (= 1.51478\ldots). \tag{7}$$

So, on "average" we can replace the product $\prod_{p|d} \frac{p-1}{p-2}$ in (6) by its mean value $1/c_2$, hence the twin constant c_2 disappears and neglecting the $\ln(2) = 0.69314\ldots$ we and up with the quadratic equation for $u = \ln(p_f(d))$:

$$u^2 - u\ln(d) - d = 0$$

The positive solution of this equation gives:

$$p_f(d) = \sqrt{d}\ \mathrm{e}^{\frac{1}{2}\sqrt{\ln^2(d)+4d}}. \tag{8}$$

The comparison of this formula with the actual available data from the computer search is shown in the Fig. 1. Most of the points plotted on this figure comes from my own search up to $2^{44} = 1.76 \times 10^{13}$ [4]. First occurences $p_f(d) > 2^{44}$ I have taken from [5] and [6] (see also [7], [8]). On the Fig.1 there is also a plot of the conjecture made by Shanks [9], who guessed that

$$p_f(d) \sim \mathrm{e}^{\sqrt{d}}, \tag{9}$$

while from (8) for large d it follows that

$$p_f(d) \sim \sqrt{d}\ \mathrm{e}^{\sqrt{d}}.$$

The inspection of the Fig.1 shows, that (9) can be regarded as a lower bound for $p_f(d)$.

Acknowledgment: I thank Prof. T. Nicely for providing me with the unpublished yet first occurences of primes gaps up to 3.6×10^{14}. The author receives support from the KBN research grant No 2 P302 057 07

References

[1] M.Wolf, "Generalized Brun's constants", preprint IFTUWr 910/97, March 1997, available in postscript file at http://www.ift.uni.wroc.pl/~mwolf/

[2] V.Brun, "La serie 1/5 + 1/7 +... est convergente ou finie", *Bull. Sci.Math.* **43** (1919), pp.124-128

[3] E. Bombieri and H. Davenport, "Small differences between prime numbers", *Proc. Royal Soc.*, **A293**, (1966), 1-18

[4] M.Wolf, "Some conjectures on the gaps between consecutive primes", preprint IFTUWr 894/95, submitted to *J.Number Theory*;

[5] J.Young and A.Potler, “First occurrence Prime Gaps”, *Math.Comp.* **52** (1989), 221-224

[6] T.Nicely, “New Maximal Prime Gaps and First Occurrences”, (in preparation).

[7] S.Weintraub, “A Prime Gap of 864”, J. of Recr. Math., **25** (1993), pp. 42-43

[8] P. Cutter, “Finding Prime Pairs with Particular Gaps”, preprint, available at http://www.math.uga.edu/~pamelar/

[9] D.Shanks, ”On Maximal Gaps between Successive Primes”, *Math.Comp.* **18** (1964), 464

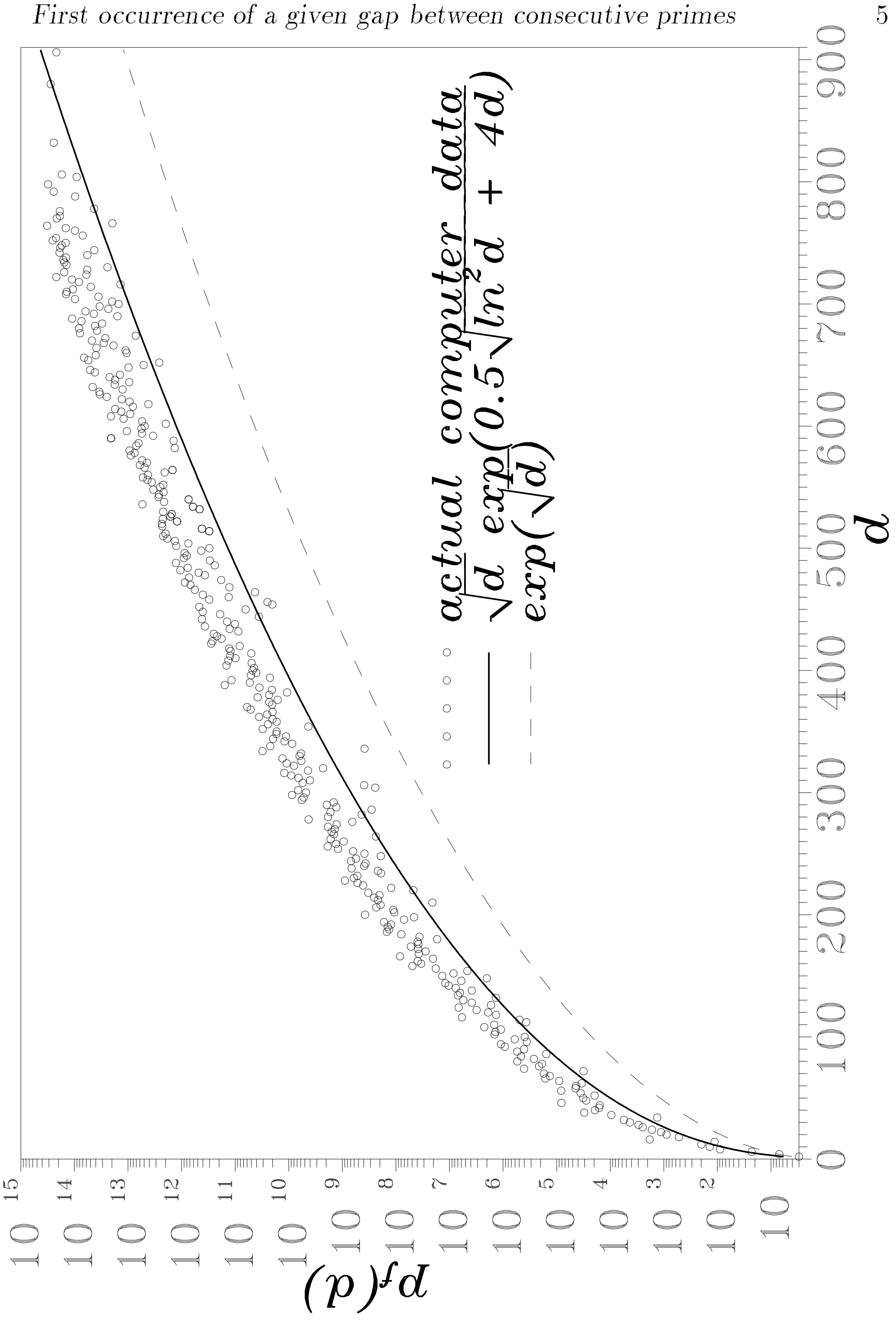

Fig.1 The plot showing the actual appearence of a given gap between consecutive primes found by a computer search (circles) and the plot of the formula (8) (solid line) as well as the conjecture (9) by dashed line.