The Zeros of the Riemann Zeta-Function

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1—It is well known that the distribution of the zeros of the Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (s = \sigma + it)$$

plays a fundamental part in the theory of prime numbers. It was conjectured by Riemann that all the complex zeros of $\zeta(s)$ lie on the line $\sigma = \frac{1}{2}$, but this hypothesis has never been proved or disproved. It is therefore natural to enquire how far the hypothesis is supported by numerical calculations.

The most extensive calculations of this kind† have been undertaken by Gram, Backlund, and Hutchinson.‡ The final result obtained by Hutchinson is that $\zeta(s)$ has 138 zeros on $\sigma = \frac{1}{2}$ between $t = 0$ and $t = 300$, and no other zeros between these values of t.

The method of all these authors seems to be substantially the same. Hutchinson uses the formula

$$\zeta(s) = \sum_{\nu=1}^{n-1} \frac{1}{\nu^s} + \frac{1}{2n^s} + \frac{n^{1-s}}{1-s} + \sum_{\nu=1}^{k} (-1)^{\nu-1} \frac{B_\nu}{(2\nu)!} \frac{s(s+1)\ldots(s+2\nu-2)}{n^{s+2\nu-1}} + R_k$$

(R_k satisfying certain inequalities), which, with suitable values of n and k, can be used to obtain arbitrarily close approximations to $\zeta(s)$ in the critical strip. The calculations which it demands are very laborious if t is at all large.

There is, however, another formula available. The well-known approximate functional equation of Hardy and Littlewood is

$$\zeta(s) = \sum_{n<x} \frac{1}{n^s} + \chi(s) \sum_{n<y} \frac{1}{n^{1-s}} + O(x^{-\sigma}) + O(y^{\sigma-1}|t|^{\frac{1}{2}-\sigma}),$$

where $2\pi xy = |t|$, and $\chi(s) = \pi^{s-\frac{1}{2}}\Gamma(\frac{1}{2} - \frac{1}{2}s)/\Gamma(\frac{1}{2}s)$. If $x = y$, each is equal to $(|t|/2\pi)^{\frac{1}{2}}$, so that for large t there are only $O(\sqrt{t})$ terms to be calculated. But the method has another advantage. It has recently been found that certain cases of the formula were known to Riemann,

† *See* my Cambridge Tract "The Zeta-Function of Riemann," § 3.13.

‡ Hutchinson, 'Trans. Amer. Math. Soc.,' vol. 27, p. 49 (1925).

§ Siegel, 'Quell. Gesch. Math.,' vol. 2, p. 45 (1931).

and that he obtained an asymptotic series instead of the above O-terms. The series proceeds in powers of $t^{-\frac{1}{2}}$, and the coefficients are trigonometrical functions, and so are suitable for calculation. Since the series is asymptotic and (presumably) not convergent, the degree of approximation obtainable for a particular t depends on the constants involved. I find that by taking the first term of the asymptotic series (of order $t^{-\frac{1}{4}}$ on $\sigma = \frac{1}{2}$) explicitly, and finding an upper bound for the remainder, we obtain a sufficiently close approximation for the purpose of showing that the zeros lie on $\sigma = \frac{1}{2}$, as far as the calculations go. I have now carried them as far as $t = 390$, and find that all the zeros up to this point lie on $\sigma = \frac{1}{2}$.

1.1—The paper is in three parts. In the first, the approximate formulae for $\zeta(s)$ are proved. There is no new principle here; but approximations which are familiar in the ordinary O-form have to be obtained with actual constants. In the second part the results of the calculations are described. I then conclude with some further theoretical considerations on the problem of the zeros.

The following notations are used in §§ 2–9. We write as usual $s = \sigma + it$, and always take $t > 0$. We put

$$t = 2\pi\tau, \qquad m = [\sqrt{\tau}], \qquad \eta = \sqrt{(2\pi t)} = 2\pi\sqrt{\tau}.$$

The various contours used are denoted by $\Gamma, \Gamma_1, \ldots$, and the integrals by $J, J_1, \ldots, K_1, \ldots, L_1, \ldots$. The complex variable of integration is $w = u + iv$, and $\lambda = |w - i\eta|$. The function $r(w)$ is defined in § 4, and $\psi = \psi(\sigma, t)$ is used temporarily in this section.

We write

$$\chi(s) = \pi^{s-\frac{1}{2}}\Gamma(\tfrac{1}{2} - \tfrac{1}{2}s)/\Gamma(\tfrac{1}{2}s), \qquad \chi(\tfrac{1}{2} + it) = e^{-2i\vartheta},$$

so that

$$\vartheta = -\tfrac{1}{2}t\log\pi + \mathbf{I}\log\Gamma(\tfrac{1}{4} + \tfrac{1}{2}it);$$

also

$$\kappa = \vartheta/(2\pi).$$

By $\omega_1, \omega_2, \ldots$, we denote remainder terms in asymptotic formulae for the Γ-function.

We write $$f(\tau) = f(t/2\pi) = e^{i\vartheta}\zeta(\tfrac{1}{2} + it),$$

$f(\tau)$ being real for real τ; also

$$g(\tau) = \frac{(-1)^{m-1}}{\tau^{\frac{1}{4}}} \frac{\cos 2\pi\{\tau - (2m+1)\sqrt{\tau} - \frac{1}{16}\}}{\cos 2\pi\sqrt{\tau}}$$

$$h(\xi) = \frac{\cos 2\pi(\xi^2 - \xi - \frac{1}{16})}{\cos 2\pi\xi},$$

so that, if $\sqrt{\tau} = m + \xi$, then $g(\tau) = (-1)^{m-1}\tau^{-\frac{1}{4}}h(\xi)$.

Also

$$\alpha_\nu = \alpha_\nu(\tau) = \nu^{-\frac{1}{2}} \cos 2\pi(\kappa - \tau \log \nu).$$

2—We begin with some lemmas, which are familiar enough, but which contain a useful statement of inequalities which we shall use.

LEMMA α—

$$\text{Arc tan} \frac{c}{1-c} - c \geqq c^2 \qquad (0 \leqq c \leqq \tfrac{1}{2}).$$

For if

$$F(c) = \text{arc tan} \frac{c}{1-c} - c - c^2,$$

then

$$F'(c) = \frac{2c^2(1-2c)}{1-2c+2c^2} \geqq 0 \qquad (0 \leqq c \leqq \tfrac{1}{2}),$$

and since $F(0) = 0$, it follows that $F(c) \geqq 0$ for $0 \leqq c \leqq \frac{1}{2}$.

LEMMA β—

$$c - \text{arc tan} \frac{c}{1+c} \geqq \frac{c^2}{1+c_0} \qquad (0 \leqq c \leqq c_0).$$

For arc tan $x \leqq x$ $(0 \leqq x < 1)$, so that

$$c - \text{arc tan} \frac{c}{1+c} \geqq c - \frac{c}{1+c} = \frac{c^2}{1+c} \geqq \frac{c^2}{1+c_0}.$$

LEMMA γ—

$$|\Gamma(\tfrac{1}{2} - it)| < (2\pi)^{\frac{1}{2}} e^{-\frac{1}{2}\pi t} \qquad (t > 0).$$

For

$$|\Gamma(\tfrac{1}{2} - it)|^2 = \pi \operatorname{sech} \pi t < 2\pi e^{-\pi t}.$$

LEMMA δ—*For* $\sigma > 0$, $t > 0$,

$$\log \Gamma(\sigma + it) = (\sigma + it - \tfrac{1}{2}) \log(\sigma + it) - \sigma - it + \tfrac{1}{2} \log 2\pi + \frac{1}{12(\sigma + it)} + \omega_1,$$

where

$$|\omega_1| < \frac{1}{720\sigma t^2}.$$

We have

$$\log \Gamma(\sigma + it) = (\sigma + it - \tfrac{1}{2}) \log(\sigma + it) - \sigma - it + \tfrac{1}{2} \log 2\pi + 2\int_0^\infty \frac{du}{e^{2\pi u} - 1} \int_0^{u/(\sigma+it)} \frac{dv}{1+v^2}.$$

We write

$$\int_0^{u/(\sigma+it)} \frac{dv}{1+v^2} = \frac{u}{\sigma+it} - \int_0^{u/(\sigma+it)} \frac{v^2}{1+v^2}\, dv,$$

and note that

$$\int_0^\infty \frac{u\, du}{e^{2\pi u} - 1} = \frac{1}{24}, \quad \int_0^\infty \frac{u^3\, du}{e^{2\pi u} - 1} = \frac{1}{240}.$$

This gives the term $\{12(\sigma + it)\}^{-1}$; and

$$\left|\int_0^{u/(\sigma+it)} \frac{v^2}{1+v^2}\, dv\right| \leqq \frac{u^3}{3|\sigma+it|^3} \max_{0 \leqq \lambda \leqq u} \frac{1}{|1 + \lambda^2(\sigma+it)^{-2}|}$$

$$= \frac{u^3}{3|\sigma+it|} \max \frac{1}{|\sigma^2 - t^2 + \lambda^2 + 2i\sigma t|} \leqq \frac{u^3}{3|\sigma+it|\, 2\sigma t} \leqq \frac{u^3}{6\sigma t^2},$$

and the inequality satisfied by ω_1 follows.

LEMMA ε—*For* $\frac{1}{2} \leqq \sigma \leqq 2$, $t \geqq 10$,

$$|\Gamma(\sigma + it)| < 1{\cdot}04\, (2\pi)^{\frac{1}{2}}\, t^{\sigma - \frac{1}{2}}\, e^{-\frac{1}{2}\pi t}.$$

By Lemma δ

$$\log|\Gamma(\sigma+it)| = -\tfrac{1}{2}\pi t + (\sigma - \tfrac{1}{2}) \log t + \tfrac{1}{2}(\sigma - \tfrac{1}{2}) \log\left(1 + \frac{\sigma^2}{t^2}\right) + t \,\text{arc tan} \frac{\sigma}{t} - \sigma + \tfrac{1}{2} \log 2\pi + \omega_2,$$

where

$$|\omega_2| \leqq \frac{1}{12|\sigma+it|} + \frac{1}{720\sigma t^2}.$$

Since $\log(1 + \sigma^2/t^2) \leqq \sigma^2/t^2$, t arc tan $(\sigma/t) \leqq \sigma$, we obtain

$$\log|\Gamma(\sigma+it)| \leqq -\tfrac{1}{2}\pi t + (\sigma - \tfrac{1}{2}) \log t + \tfrac{1}{2} \log 2\pi + \omega_3,$$

where

$$|\omega_3| \leqq \frac{(\sigma - \frac{1}{2})\sigma^2}{2t^2} + \frac{1}{12|\sigma+it|} + \frac{1}{720\sigma t^2} < \frac{3}{t^2} + \frac{1}{12t} + \frac{1}{360t^2} < 0{\cdot}039 < \log 1{\cdot}04.$$

LEMMA ζ—*For* $0 < \sigma \leqq 2$, $t \geqq 4$,

$$\log \Gamma(\sigma + it) = (\sigma + it - \tfrac{1}{2}) \log it - it + \tfrac{1}{2} \log 2\pi + \frac{\sigma^2 - \sigma}{2it} + \frac{1}{12(\sigma+it)} + \omega_4,$$

where

$$|\omega_4| < \frac{\sigma^2|\sigma - \frac{1}{2}|}{2t^2} + \frac{\sigma^3}{t^2} + \frac{1}{720\sigma t^2}.$$

We have

$$(\sigma + it - \tfrac{1}{2}) \log (\sigma + it) = (\sigma + it - \tfrac{1}{2}) \left\{ \log it + \frac{\sigma}{it} - \tfrac{1}{2} \frac{\sigma^2}{(it)^2} \right\} + \omega_5,$$

where

$$|\omega_5| \leq \frac{|\sigma + it - \tfrac{1}{2}|\, \sigma^3}{3(1 - \sigma/t)\, t^3} \leq \frac{\sigma^3}{t^2}$$

in the given region. The result now easily follows from Lemma δ.

Immediate consequences are:

(i) $$\log \Gamma(\tfrac{1}{2} - it) = - it \log(-it) + it + \tfrac{1}{2} \log 2\pi + \frac{1}{24it} + \omega_6,$$

where

$$\left| \omega_6 \frac{61}{360\, t^2} \right| < \frac{7}{40 t^2};$$

(ii) $$\mathrm{I} \log \Gamma(\tfrac{1}{4} + \tfrac{1}{2}it) = \tfrac{1}{2}t \log \tfrac{1}{2}t - \tfrac{1}{2}t - \tfrac{1}{8}\pi + \frac{1}{48t} + \omega_7,$$

where

$$|\omega_7| < \frac{167}{1440 t^2} + \frac{1}{24 t^3} < \frac{2}{15 t^2}.$$

3—By a well-known series of transformations

$$\zeta(s) = \sum_{\nu=1}^{m} \frac{1}{\nu^s} - \frac{1}{\Gamma(s)} \int_0^\infty \frac{w^{s-1} e^{-mw}}{e^w - 1}\, dw$$

$$= \sum_{\nu=1}^{m} \frac{1}{\nu^s} - \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-w)^{s-1} e^{-mw}}{e^w - 1}\, dw$$

$$= \sum_{\nu=1}^{m} \frac{1}{\nu^s} + \chi(s) \sum_{\nu=1}^{m} \frac{1}{\nu^{1-s}} - \frac{\Gamma(1-s)}{2\pi i} \int_{C'} \frac{(-w)^{s-1} e^{-mw}}{e^w - 1}\, dw,$$

where C is a loop coming from infinity on the positive real axis, encircling the origin in the positive direction, but excluding the poles of the integrand, and returning to infinity; and C′ is a similar contour including the first m poles of the integrand on each side of the real axis; and

$$(-w)^{s-1} = e^{(s-1)\{\log |w| + i\, am(-w)\}}$$

where $am(-w)$ increases from $-\pi$ to π round the loop. The final formula holds for all values of s except $s = 1$.

We next deform the contour C′ as follows. Take the contour from infinity along the straight line $v = u + \eta$ as far as $u = -\tfrac{1}{2}\eta$, then along $u = -\tfrac{1}{2}\eta$ as far as $v = -(2m+1)\pi$; and then along $v = -(2m+1)\pi$ back to infinity. Since this might pass through a pole, we have to make a further deformation. The part of the contour outside the half-strip

$v > 0$, $|u| \leq \tfrac{1}{2}\pi$ is left as it is. We cross the half-strip along whichever of the lines $v = 2\pi(\sqrt{\tau} \pm \tfrac{1}{4})$ is further from any of the lines $v = 2\pi r$, where r is an integer, and join up along $u = -\tfrac{1}{2}\pi$ or $u = \tfrac{1}{2}\pi$ as the case may be.

To make the argument definite, suppose that $\sqrt{\tau} + \tfrac{1}{4}$ is farther from an integer than $\sqrt{\tau} - \tfrac{1}{4}$. The contour, Γ say, then consists of nine parts, Γ_1 to Γ_9, which are the straight lines joining the following points:—

$$\infty e^{\frac{1}{4}i\pi}, \quad i\eta + \eta t^{-\frac{1}{2}} e^{\frac{1}{4}i\pi}, \quad i\eta + 2^{-\frac{1}{2}}\pi e^{\frac{1}{4}i\pi}, \quad i\eta + \tfrac{1}{2}i\pi,$$

$$i\eta + 2^{-\frac{1}{2}}\pi e^{\frac{3}{4}i\pi}, \quad i\eta - 2^{-\frac{1}{2}}\pi e^{\frac{1}{4}i\pi}, \quad i\eta - \eta t^{-\frac{1}{2}} e^{\frac{1}{4}i\pi},$$

$$i\eta - 2^{-\frac{1}{2}}\eta e^{\frac{1}{4}i\pi}, \quad -\tfrac{1}{2}\eta - (2m+1)\, i\pi, \quad +\infty.$$

Let

$$J = J(\sigma) = \int_\Gamma \frac{(-w)^{s-1} e^{-mw}}{e^w - 1}\, dw = \sum_{k=1}^{9} \int_{\Gamma_k} = \sum_{k=1}^{9} J_k;$$

and put

$$\sum_{k=2}^{6} J_k = \int_{\Gamma_2 + \dots + \Gamma_6} \{(-w)^{s-1} - (-i\eta)^{s-1} e^{(w-i\eta)\sqrt{\tau} + \frac{1}{4}i(w-i\eta)^2/\pi}\} \frac{e^{-mw}}{e^w - 1}\, dw$$

$$+ \int_{\Gamma_1 + \dots + \Gamma_6 + \Gamma'_7} (-i\eta)^{s-1} \frac{e^{(w-i\eta)\sqrt{\tau} + \frac{1}{4}i(w-i\eta)^2/\pi - mw}}{e^w - 1}\, dw - \int_{\Gamma_1} - \int_{\Gamma'_7}$$

$$= \sum_{k=2}^{6} K_k + L_1 - L_2 - L_3,$$

the integrands in L_1, L_2, and L_3 being the same, and Γ'_7 being Γ_7 continued to infinity. Thus

$$J = J_1 + K_2 + K_3 + K_4 + K_5 + K_6 + J_7 + J_8 + J_9 + L_1 - L_2 - L_3. \tag{3.1}$$

The main term is L_1. Let L be a straight line parallel to $u = v$, intersecting the imaginary axis between 0 and $2\pi i$. Then, as in Siegel's paper referred to above,

$$\int_L \frac{e^{aw + \frac{1}{4}iw^2/\pi}}{e^w - 1}\, dw = 2\pi \frac{\cos \pi(\tfrac{1}{2}a^2 - a - \tfrac{1}{8})}{\cos \pi a}\, e^{i\pi(\frac{1}{4}a^2 - \frac{1}{8})}.$$

Hence, putting $a = 2(\sqrt{\tau} - m)$,

$$L_1 = -(-i\eta)^{s-1} \int_L \frac{e^{(w + 2m\pi i - i\eta)\sqrt{\tau} + \frac{1}{4}i(w + 2m\pi i - i\eta)^2/\pi - mw}}{e^w - 1}\, dw$$

$$= (-1)^{m+1} (-i\eta)^{s+1} e^{-\frac{1}{2}it - \frac{1}{8}\pi i} \,.\, 2\pi \frac{\cos 2\pi\{\tau - (2m+1)\sqrt{\tau} - \tfrac{1}{16}\}}{\cos 2\pi\sqrt{\tau}}$$

$$= e^{-\frac{1}{2}i\pi(s - \frac{1}{2})} (2\pi)^{\frac{1}{2}s + \frac{1}{2}} t^{\frac{1}{2}s - \frac{1}{2}} e^{-\frac{1}{2}it} g(\tau). \tag{3.2}$$

4—We next observe that

$$(-w)^{s-1} = (-i\eta)^{s-1}\, e^{(s-1)\log\{w/(i\eta)\}},$$

and

$$(\sigma - 1 + it)\log\frac{w}{i\eta} = (\sigma - 1 + it)\left\{\frac{w-i\eta}{i\eta} - \frac12\left(\frac{w-i\eta}{i\eta}\right)^2 + \frac13\left(\frac{w-i\eta}{i\eta}\right)^3 - \dots\right\}$$

$$= (w - i\eta)\sqrt{\tau} + \frac{i}{4\pi}(w-i\eta)^2 + \dots .$$

We therefore write

$$(-w)^{s-1} = (-i\eta)^{s-1}\, e^{(w-i\eta)\sqrt{\tau}+\frac14 i(w-i\eta)^2/\pi}\{1 + r(w)\},$$

where

$$r(w) = \exp\left[it\left\{\frac13\left(\frac{w-i\eta}{i\eta}\right)^3 - \dots\right\} + (\sigma-1)\left\{\frac{w-i\eta}{i\eta} - \frac12\left(\frac{w-i\eta}{i\eta}\right)^2 + \dots\right\}\right] - 1.$$

Then

$$K_2 = \int_{\Gamma_2}(-i\eta)^{s-1}\frac{e^{(w-i\eta)\sqrt{\tau}+\frac14 i(w-i\eta)^2/\pi - mw}}{e^w - 1}\, r(w)\, dw.$$

For $|z| \leqq \frac32$

$$|e^z - 1| = \left|z + \frac{z^2}{2!} + \dots\right| \leqq |z| + \frac{|z|^2}{2} + \frac{|z|^2}{2^2} + \dots = \frac{|z|}{1 - \frac12|z|}.$$

Hence, putting $|w - i\eta| = \lambda$,

$$|r(w)| \leqq \frac{t\left|\frac13\left(\frac{w-i\eta}{i\eta}\right)^3 - \dots\right| + |1-\sigma|\left|\frac{w-i\eta}{i\eta} - \dots\right|}{1 - \frac12 t\left|\frac13\left(\frac{w-i\eta}{i\eta}\right)^3 - \dots\right| - \frac12|1-\sigma|\left|\frac{w-i\eta}{i\eta} - \dots\right|}$$

$$\leqq \frac{\left\{\frac13 t\left(\frac{\lambda}{\eta}\right)^3 + |1-\sigma|\frac{\lambda}{\eta}\right\}\Big/\left(1-\frac{\lambda}{\eta}\right)}{1 - \frac12\left\{\frac13 t\left(\frac{\lambda}{\eta}\right)^3 + |1-\sigma|\frac{\lambda}{\eta}\right\}\Big/\left(1-\frac{\lambda}{\eta}\right)} = \frac{\frac13 t\left(\frac{\lambda}{\eta}\right)^3 + |1-\sigma|\frac{\lambda}{\eta}}{1 - \frac16 t\left(\frac{\lambda}{\eta}\right)^3 - \left(1+\frac12|1-\sigma|\right)\frac{\lambda}{\eta}}.$$

On Γ_2, $(\lambda/\eta)^3 \leqq 1/t$, so that we also have

$$|r(w)| \leqq \frac{\frac13 t(\lambda/\eta)^3 + |1-\sigma|\,\lambda/\eta}{\frac56 - \left(1+\frac12|1-\sigma|\right)t^{-\frac13}}.$$

Also

$$\mathbf{R}\left\{(w-i\eta)\sqrt{\tau} + \frac{i}{4\pi}(w-i\eta)^2 - (m+1)w\right\}$$

$$= \frac{\lambda\sqrt{\tau}}{\sqrt2} - \frac{\lambda^2}{4\pi} - (m+1)\frac{\lambda}{\sqrt2} \leqq -\frac{\lambda^2}{4\pi},$$

and $|1 - e^{-w}| \geqq 1 - e^{-\frac12\pi}$. Hence

$$|K_2| \leqq \frac{e^{\frac12\pi t}}{(2\pi t)^{\frac12-\frac12\sigma}(1-e^{-\frac12\pi})}\int_{\pi/\sqrt2}^{\infty}\frac{\frac13 t(\lambda/\eta)^3 + |1-\sigma|\,\lambda/\eta}{\frac56 - \left(1+\frac12|1-\sigma|\right)t^{-\frac13}}\, e^{-\frac14\lambda^2/\pi}\, d\lambda$$

$$= \frac{e^{\frac12\pi t - \frac18\pi}}{(2\pi t)^{1-\frac12\sigma}(1-e^{-\frac12\pi})}\cdot\frac{\frac16\pi^2 + \frac43\pi + 2\pi|1-\sigma|}{\frac56 - \left(1+\frac12|1-\sigma|\right)t^{-\frac13}}. \quad (4.1)$$

The same inequality holds for K_6.

Next consider K_3. Here $\lambda \leqq \pi/\sqrt2$, so that

$$|r(w)| \leqq \frac{\frac13 t\left(\frac{\sqrt\pi}{2\sqrt t}\right)^3 + |1-\sigma|\frac{\sqrt\pi}{2\sqrt t}}{1 - \frac16 t\left(\frac{\sqrt\pi}{2\sqrt t}\right)^3 - \left(1+\frac12|1-\sigma|\right)\frac{\sqrt\pi}{2\sqrt t}}.$$

Denote the denominator on the right-hand side by $\psi = \psi(\sigma, t)$.

Also $w = u + i(\eta \pm \frac12\pi)$,

$$\mathbf{R}\{(w-i\eta)\sqrt\tau + \tfrac14 i(w-i\eta)^2/\pi - (m+1)w\}$$

$$= u\sqrt\tau \mp \tfrac14 u - (m+1)u \leqq \tfrac14 u,$$

(allowing for the two possible figures); and

$$|1 - e^{-w}|^2 = 1 - 2e^{-u}\cos v + e^{-2u} \geqq 1,$$

since here $\cos v \leqq 0$. Hence

$$|K_3| \leqq \frac{e^{\frac12\pi t}}{(2\pi t)^{\frac12-\frac12\sigma}}\left(\frac{\pi^{\frac32}}{24} + \frac{\pi^{\frac12}|1-\sigma|}{2}\right)\frac{1}{\psi\sqrt t}\int_0^{\frac12\pi} e^{\frac14 u}\, du. \quad (4.2)$$

The same result holds for K_4.

Next consider K_5. We have the same inequality for $r(w)$ as in the previous case. Also $w = -\frac12\pi + iv$, so that

$$\mathbf{R}\{(w-i\eta)\sqrt\tau + \tfrac14 i(w-i\eta)^2/\pi - mw\}$$

$$= -\tfrac12\pi\sqrt\tau + \tfrac14(v-\eta) + \tfrac12 m\pi \leqq \tfrac18\pi;$$

and

$$|e^w - 1| \geqq 1 - e^{-\frac12\pi}.$$

Hence

$$|K_5| \leqq \frac{e^{\frac12\pi t}}{(2\pi t)^{\frac12-\frac12\sigma}}\frac{\pi e^{\frac18\pi}}{1-e^{-\frac12\pi}}\left(\frac{\pi^{\frac32}}{24} + \frac{\pi^{\frac12}|1-\sigma|}{2}\right)\frac{1}{\psi\sqrt t}. \quad (4.3)$$

Next consider L_2. Arguing as for K_2, we have

$$|L_2| \leqq \frac{e^{\frac12\pi t}}{(2\pi t)^{\frac12-\frac12\sigma}}\int_{\eta t^{-\frac13}}^{\infty}\frac{e^{-\frac14\lambda^2/\pi}}{1 - \exp(-\eta t^{-\frac13}/\sqrt2)}\, d\lambda.$$

Now

$$\int_{\eta t^{-\frac{1}{2}}}^{\infty} e^{-\frac{1}{2}\lambda^2/\pi}\, d\lambda = \int_{\frac{1}{2}t^{\frac{1}{2}}}^{\infty} e^{-\mu}\left(\frac{\pi}{\mu}\right)^{\frac{1}{2}} d\mu < \left(\frac{2\pi}{t^{\frac{1}{2}}}\right)^{\frac{1}{2}} e^{-\frac{1}{2}t^{\frac{1}{2}}}.$$

Hence

$$|L_2| \leqq \frac{e^{\frac{1}{2}\pi t}}{(2\pi t)^{\frac{1}{2}-\frac{1}{2}\sigma}} \frac{e^{-\frac{1}{2}t^{\frac{1}{2}}}}{1 - e^{-\pi^{\frac{1}{2}}t^{\frac{1}{2}}}} \left(\frac{2\pi}{t^{\frac{1}{2}}}\right)^{\frac{1}{2}}. \tag{4.4}$$

The same result holds for L_3.

Next consider J_1, and suppose first that $\sigma \leqq 1$. Here $w = i\eta + \lambda e^{\frac{1}{4}i\pi}$,

$$(-w)^{s-1} = \exp\left[(\sigma - 1 + it)\{\log(\eta + \lambda e^{-\frac{1}{4}i\pi}) - \tfrac{1}{2}i\pi\}\right],$$

and

$$\log(\eta + \lambda e^{-\frac{1}{4}i\pi}) = \tfrac{1}{2}\log(\eta^2 + \sqrt{2}\eta\lambda + \lambda^2) - i \arctan\frac{\lambda}{\eta\sqrt{2} + \lambda}.$$

Hence

$$|(-w)^{s-1}| = (\eta^2 + \eta\lambda\sqrt{2} + \lambda^2)^{\frac{1}{2}\sigma-\frac{1}{2}}\, e^{\frac{1}{2}\pi t + t\arctan\{\lambda/(\eta\sqrt{2}+\lambda)\}}$$
$$\leqq \eta^{\sigma-1}\, e^{\frac{1}{2}\pi t + t\arctan\{\lambda/(\eta\sqrt{2}+\lambda)\}}.$$

Let J_1' be the part of J_1 with $\lambda \leqq \eta/\sqrt{2}$, J_1'' the remainder. In J_1', by Lemma β,

$$(m+1)\frac{\lambda}{\sqrt{2}} - t\arctan\frac{\lambda}{\eta\sqrt{2}+\lambda} \geqq t\left(\frac{\lambda}{\eta\sqrt{2}} - \arctan\frac{\lambda}{\eta\sqrt{2}+\lambda}\right)$$
$$\geqq \frac{t\lambda^2}{2\eta^2(1+\frac{1}{4})} = \frac{\lambda^2}{5\pi}.$$

Hence

$$|J_1'| \leqq \frac{e^{\frac{1}{2}\pi t}}{(2\pi t)^{\frac{1}{2}-\frac{1}{2}\sigma}(1 - e^{-\pi^{\frac{1}{2}}t^{\frac{1}{2}}})} \int_{\eta t^{-\frac{1}{2}}}^{\infty} e^{-\frac{1}{2}\lambda^2/\pi}\, d\lambda.$$

This integral is

$$\int_{\lambda=\eta t^{-\frac{1}{2}}}^{\infty} e^{-\mu}\frac{5\pi}{2\lambda}\, d\mu \leqq \frac{5\pi}{2\eta t^{-\frac{1}{2}}} e^{-\frac{1}{2}\eta^2 t^{-1}/\pi} = \frac{5\pi t^{\frac{1}{2}}}{2(2\pi t)^{\frac{1}{2}}} e^{-\frac{1}{2}t^{\frac{1}{2}}}.$$

Hence

$$|J_1'| \leqq \frac{5\pi t^{\frac{1}{2}} e^{\frac{1}{2}\pi t - \frac{1}{2}t^{\frac{1}{2}}}}{2(2\pi t)^{1-\frac{1}{2}\sigma}(1 - e^{-\pi^{\frac{1}{2}}t^{\frac{1}{2}}})}. \tag{4.5}$$

In J_1'', $\eta^2 + \eta\lambda\sqrt{2} + \lambda^2 \geqq \frac{5}{2}\eta^2$; hence

$$|J_1''| \leqq \frac{e^{\frac{1}{2}\pi t}}{(\frac{5}{2}\eta^2)^{\frac{1}{2}-\frac{1}{2}\sigma}(1 - e^{-\pi^{\frac{1}{2}}t^{\frac{1}{2}}})} \int_{\eta/\sqrt{2}}^{\infty} \exp\left\{t\left(\arctan\frac{\lambda}{\eta\sqrt{2}+\lambda} - \frac{\lambda}{\eta\sqrt{2}}\right)\right\} d\lambda.$$

If

$$\mu = \frac{\lambda}{\eta\sqrt{2}} - \arctan\frac{\lambda}{\eta\sqrt{2}+\lambda},$$

then

$$\frac{d\mu}{d\lambda} = \frac{1}{\eta\sqrt{2}} - \frac{\eta\sqrt{2}}{2(\eta^2 + \eta\lambda\sqrt{2} + \lambda^2)},$$

which is positive and steadily increasing. Hence the above integral is

$$\int_{\mu_0}^{\infty} e^{-\mu t}\frac{d\lambda}{d\mu}\, d\mu \leqq \left(\frac{d\lambda}{d\mu}\right)_0 \int_{\mu_0}^{\infty} e^{-\mu t}\, d\mu = \left(\frac{d\lambda}{d\mu}\right)_0 \frac{e^{-\mu_0 t}}{t},$$

where

$$\mu_0 = \tfrac{1}{2} - \arctan\tfrac{1}{3} \geqq \tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6},$$

and

$$\left(\frac{d\mu}{d\lambda}\right)_0 = \frac{3}{5\eta\sqrt{2}}.$$

Hence

$$|J_1''| \leqq \frac{5\eta\sqrt{2}\, e^{\frac{1}{2}\pi t - \frac{1}{6}t}}{3t(\frac{5}{2}\eta^2)^{\frac{1}{2}-\frac{1}{2}\sigma}(1 - e^{-\pi^{\frac{1}{2}}t^{\frac{1}{2}}})}. \tag{4.6}$$

In J_7, $w = i\eta - \lambda e^{\frac{1}{4}i\pi}$, and

$$|(-w)^{s-1} e^{-mw}| = (\eta^2 - \eta\lambda\sqrt{2} + \lambda^2)^{\frac{1}{2}\sigma-\frac{1}{2}}$$
$$\times \exp\left\{t\left(\tfrac{1}{2}\pi + \arctan\frac{\lambda}{\lambda - \eta\sqrt{2}}\right) + \frac{m\lambda}{\sqrt{2}}\right\}$$
$$\leqq (\tfrac{1}{2}\eta^2)^{\frac{1}{2}\sigma-\frac{1}{2}} \exp\left\{t\left(\tfrac{1}{2}\pi + \arctan\frac{\lambda}{\lambda - \eta\sqrt{2}} + \frac{\lambda}{\eta\sqrt{2}}\right)\right\}$$
$$\leqq (\tfrac{1}{2}\eta^2)^{\frac{1}{2}\sigma-\frac{1}{2}}\, e^{t(\frac{1}{2}\pi - \frac{1}{4}\lambda^2/\eta^2)}.$$

Hence

$$|J_7| \leqq \frac{e^{\frac{1}{2}\pi t}}{(\frac{1}{2}\eta^2)^{\frac{1}{2}-\frac{1}{2}\sigma}} \frac{1}{1 - e^{-\pi^{\frac{1}{2}}t^{\frac{1}{2}}}} \int_{\eta t^{-\frac{1}{2}}}^{\infty} e^{-\frac{1}{2}\lambda^2/\pi}\, d\lambda,$$

which is the same as for L_2, but with an additional factor $2^{\frac{1}{2}-\frac{1}{2}\sigma}$. Hence

$$|J_7| \leqq \frac{e^{\frac{1}{2}\pi t}}{(\pi t)^{\frac{1}{2}-\frac{1}{2}\sigma}} \frac{e^{-\frac{1}{2}t^{\frac{1}{2}}}}{1 - e^{-\pi^{\frac{1}{2}}t^{\frac{1}{2}}}} \left(\frac{2\pi}{t^{\frac{1}{2}}}\right)^{\frac{1}{2}}. \tag{4.7}$$

Next consider J_8. Here $w = -\frac{1}{2}\eta + iv$, and

$$|(-w)^{s-1}| = (\tfrac{1}{4}\eta^2 + v^2)^{\frac{1}{2}\sigma-\frac{1}{2}}\, e^{t\arctan(2v/\eta)}.$$

Hence

$$|J_8| \leqq \frac{e^{\frac{1}{2}m\eta}}{1 - e^{-\frac{1}{2}\eta}} \int_{-(2m+1)\pi}^{\frac{1}{2}\eta} (\tfrac{1}{4}\eta^2 + v^2)^{\frac{1}{2}\sigma-\frac{1}{2}}\, e^{t\arctan(2v/\eta)}\, dv.$$

Let

$$\arctan\frac{2v}{\eta} = \xi, \quad \frac{d\xi}{dv} = \frac{\frac{1}{2}\eta}{\frac{1}{4}\eta^2 + v^2}.$$

Then the integral is

$$\frac{2}{\eta}\int^{\frac{1}{4}\pi} (\tfrac{1}{4}\eta^2 + v^2)^{\frac{1}{2}\sigma+\frac{1}{2}}\, e^{t\xi}\, d\xi < \frac{2}{\eta}\{\tfrac{1}{4}\eta^2 + (2m+1)^2\pi^2\}^{\frac{1}{2}\sigma+\frac{1}{2}} \frac{e^{\frac{1}{4}\pi t}}{t}.$$

We have $m\eta \leqq t$; and, if $t \geqq 2\pi$,

$$2m+1 \leqq \sqrt{(2t/\pi)} + 1 \leqq 3\sqrt{(t/2\pi)}.$$

Hence

$$|J_8| \leqq \frac{2\{\frac{1}{4}\eta^2 + (2m+1)^2\pi^2\}^{\frac{1}{2}\sigma+\frac{1}{2}}}{\eta t(1-e^{-\frac{1}{2}\eta})} e^{t(\frac{1}{3}+\frac{1}{4}\pi)} \leqq \frac{2(5\pi t)^{\frac{1}{2}\sigma+\frac{1}{2}}}{\eta t(1-e^{-\pi^{\frac{1}{2}}t^{\frac{1}{2}}})} e^{t(\frac{1}{3}+\frac{1}{4}\pi)}. \quad (4.8)$$

Finally, on Γ_9,

$$|(-w)^{s-1}| = |w|^{\sigma-1} \exp[-\tfrac{1}{2}\pi t - t \arctan\{u/(2m+1)\pi\}].$$

Hence, if $\sigma \leqq 1$,

$$|J_9| \leqq \frac{e^{-\frac{1}{2}\pi t}}{\{(2m+1)\pi\}^{1-\sigma}} \int_{-\frac{1}{2}\eta}^{\infty} \exp\left\{-t \arctan\frac{u}{(2m+1)\pi} - mu\right\}\frac{du}{1+e^u}.$$

Here

$$\int_0^\infty \leqq \int_0^\infty e^{-(m+1)u}\,du = \frac{1}{m+1},$$

and

$$\int_{-\frac{1}{2}\eta}^{0} \leqq \int_0^{\frac{1}{2}\eta} \exp\left\{\frac{tu}{(2m+1)\pi} + mu\right\}du.$$

Now

$$\frac{t}{(2m+1)\pi} < \frac{2(m+1)^2\pi}{(2m+1)\pi} < m+2.$$

Hence

$$\int_{-\frac{1}{2}\eta}^{0} < \int_0^{\frac{1}{2}\eta} e^{(2m+2)u}\,du < \frac{e^{(m+1)\eta}}{2m+2} \leqq \frac{e^{2t}}{2m+2} \qquad (t \geqq 2\pi).$$

Hence

$$|J_9| \leqq \frac{e^{-\frac{1}{2}\pi t}(1+\frac{1}{2}e^{2t})}{\{(2m+1)\pi\}^{1-\sigma}(m+1)} < \frac{e^{(2-\frac{1}{2}\pi)t}}{\{(2m+1)\pi\}^{1-\sigma}(m+1)}. \quad (4.9)$$

5—Slight modifications are needed if $1 < \sigma \leqq 2$. In J_1',

$$\eta^2 + \eta\lambda\sqrt{2} + \lambda^2 \leqq \tfrac{5}{2}\eta^2,$$

and the right-hand side of (4.5) must be multiplied by $(\frac{5}{2})^{\sigma-1}$. In J_1''

$$\eta^2 + \eta\lambda\sqrt{2} + \lambda^2 \leqq 5\lambda^2,$$

and

$$\int_{\mu_0}^\infty e^{-\mu t}\lambda^{\sigma-1}\,d\mu = \frac{e^{\mu_0 t}}{t}\cdot\left(\frac{\eta}{\sqrt{2}}\right)^{\sigma-1} + \frac{1}{t}\int_{\mu_0}^\infty e^{-\mu t}(\sigma-1)\lambda^{\sigma-2}\frac{d\lambda}{d\mu}\,d\mu$$

$$\leqq \frac{e^{-\mu_0 t}}{t}\left(\frac{\eta}{\sqrt{2}}\right)^{\sigma-1} + \frac{\sigma-1}{t}\cdot\left(\frac{d\lambda}{d\mu}\right)_0\left(\frac{\eta}{\sqrt{2}}\right)^{\sigma-2}\cdot\frac{e^{-\mu_0 t}}{t}$$

$$\leqq \frac{e^{-\mu_0 t}}{t}\left(\frac{\eta}{\sqrt{2}}\right)^{\sigma-1}\left(1+\frac{3}{10\pi t^2}\right) < 2\frac{e^{-\mu_0 t}}{t}\left(\frac{\eta}{\sqrt{2}}\right)^{\sigma-1} \qquad (t > 1).$$

Hence the right-hand side of (4.6) must be multiplied by 2.

In J_7, $\eta^2 - \eta\lambda\sqrt{2} + \lambda^2 \leqq \eta^2$ if $\lambda \leqq \eta\sqrt{2}$. The part for which this holds gives the same upper bound as before, multiplied by $2^{\frac{1}{2}\sigma-\frac{1}{2}}$. In the remainder,

$$(\eta^2 - \eta\lambda\sqrt{2} + \lambda^2)^{\frac{1}{2}\sigma-\frac{1}{2}} \leqq \lambda^{\sigma-1} \leqq \lambda,$$

and we get an additional term

$$\frac{e^{\frac{1}{2}\pi t}}{1-e^{-\pi^{\frac{1}{2}}t^{\frac{1}{2}}}}\int_{\eta\sqrt{2}}^\infty e^{-\frac{1}{2}\lambda^2/\pi}\lambda\,d\lambda = \frac{2\pi e^{(\frac{1}{2}\pi-1)t}}{1-e^{-\pi^{\frac{1}{2}}t^{\frac{1}{2}}}}.$$

In J_9, $|w| \leqq \max[\{\frac{1}{4}\eta^2 + (2m+1)^2\pi^2\}^{\frac{1}{2}}, \sqrt{2}(2m+1)\pi] \leqq 3\sqrt{(\pi t)}$ if $\frac{1}{2}\eta \leqq u \leqq (2m+1)\pi$, and $(2m+1)\pi$ in (4.9) has to be replaced by this. If $u > (2m+1)\pi$, then $|w| < \sqrt{2u}$, and we get an additional term

$$e^{-\frac{1}{2}\pi t}\int_{(2m+1)\pi}^\infty (\sqrt{2u})^{\sigma-1}e^{-(m+1)u}\,du < e^{-\frac{1}{2}\pi t}\int_0^\infty \sqrt{2u}\,e^{-(m+1)u}\,du = \frac{e^{-\frac{1}{2}\pi t}\sqrt{2}}{(m+1)^2}.$$

Collecting together the above results, we obtain

Theorem 1—*For* $\frac{1}{2} \leqq \sigma \leqq 1$, $\quad t \geqq 8$,

$$\zeta(s) = \sum_{\nu=1}^{m}\left\{\frac{1}{\nu^s} + \frac{\chi(s)}{\nu^{1-s}}\right\} - \frac{\Gamma(1-s)}{2\pi i}e^{-\frac{1}{2}\pi t(s+\frac{1}{2})-\frac{1}{2}it}(2\pi)^{\frac{1}{2}s+\frac{1}{4}}t^{\frac{1}{2}s-\frac{1}{4}}g(\tau) + R(s),$$

where

$$|R(s)| \leqq \frac{|\Gamma(1-s)|}{2\pi}\frac{e^{\frac{1}{2}\pi t}}{t^{1-\frac{1}{2}\sigma}}\left\{\frac{e^{-\frac{1}{2}\pi}}{(2\pi)^{1-\frac{1}{2}\sigma}(1-e^{-\frac{1}{2}\pi})}\frac{\frac{1}{3}\pi^2 + 8\pi + 4\pi|1-\sigma|}{\frac{5}{6}-(1+\frac{1}{2}|1-\sigma|)t^{-\frac{1}{3}}}\right.$$

$$\left. + \frac{1}{(2\pi)^{\frac{1}{2}-\frac{1}{2}\sigma}\psi}\left(\frac{\pi^{\frac{3}{2}}}{12} + \pi^{\frac{1}{2}}|1-\sigma|\right)\left(4e^{\frac{1}{2}\pi} - 4 + \frac{\pi e^{\frac{1}{2}\pi}}{2(1-e^{-\frac{1}{2}\pi})}\right)\right\}$$

$$+ \frac{|\Gamma(1-s)|}{2\pi}\frac{e^{\frac{1}{2}\pi t}}{1-\exp(-\pi^{\frac{1}{2}}t^{\frac{1}{2}})}\left\{\frac{(2\pi)^{\frac{1}{2}\sigma}(2+2^{\frac{1}{2}-\frac{1}{2}\sigma})}{t^{\frac{1}{2}-\frac{1}{2}\sigma}}e^{-\frac{1}{2}t^{\frac{1}{3}}}\right.$$

$$\left. + \frac{5\pi t^{\frac{1}{3}}}{2(2\pi t)^{1-\frac{1}{2}\sigma}}e^{-\frac{1}{4}t^{\frac{1}{3}}} + \frac{10\pi^{\frac{1}{2}}}{3t^{\frac{1}{3}}(5\pi t)^{\frac{1}{2}-\frac{1}{2}\sigma}}e^{-\frac{1}{8}t} + \frac{2(5\pi t)^{\frac{1}{2}\sigma+\frac{1}{2}}}{t(2\pi t)^{\frac{1}{2}}}e^{\frac{1}{4}t(1-\pi)}\right\}$$

$$+ \frac{|\Gamma(1-s)|e^{(2-\frac{1}{2}\pi)t}}{2\pi\{(2m+1)\pi\}^{1-\sigma}(m+1)}.$$

The result is also true for $1 < \sigma \leqq 2$ *if the second curly bracket is multiplied by* $\frac{5}{2}$ *and the last term is replaced by*

$$\frac{|\Gamma(1-s)|}{2\pi}\left\{\frac{2\pi e^{(\frac{1}{2}\pi-1)t}}{1-\exp(-\pi^{\frac{1}{2}}t^{\frac{1}{2}})} + \frac{3^{\sigma-1}(\pi t)^{\frac{1}{2}\sigma-\frac{1}{2}}}{m+1}e^{(2-\frac{1}{2}\pi)t} + \frac{2^{\frac{1}{2}}}{(m+1)^2}e^{-\frac{1}{2}\pi t}\right\}.$

6—We now take $\sigma = \frac{1}{2}$, multiply by $e^{i\vartheta}$, and express everything in terms of τ. The left-hand side becomes $f(\tau)$. The first term on the right-hand side gives

$$2 \sum_{\nu=1}^{m} \frac{\cos(\vartheta - t \log \nu)}{\nu^{\frac{1}{2}}} = 2 \sum_{\nu=1}^{m} \frac{\cos 2\pi(\kappa - \tau \log \nu)}{\nu^{\frac{1}{2}}}.$$

In the next term,

$$e^{i\vartheta}\,\Gamma(\tfrac{1}{2} - it) = \pi^{-\frac{1}{2}it}\{\Gamma(\tfrac{1}{4} + \tfrac{1}{2}it)/\Gamma(\tfrac{1}{4} - \tfrac{1}{2}it)\}^{\frac{1}{2}}\,\Gamma(\tfrac{1}{2} - it)$$
$$= \pi^{-\frac{1}{2}it}(2\pi)^{\frac{1}{2}} e^{-\frac{1}{2}\pi t + \frac{1}{2}it - \frac{1}{4}i\pi}(2t)^{-\frac{1}{2}it} \exp\left(-\frac{7i}{48t} + \omega_6 + i\omega_7\right),$$

by the corollaries to Lemma ζ. Hence this term gives

$$g(\tau) \exp\left(-\frac{7i}{48t} + \omega_6 + i\omega_7\right).$$

In the inequality for R (s) we use Lemma γ, replace all constants by their numerical values, and observe that for $\tau \geqq 8$, *i.e.*, $t \geqq 16\pi$,

$$\exp(\pi^{\frac{1}{2}} t^{\frac{1}{2}}) \geqq \exp(2\pi)^{\frac{1}{2}} > 30.$$

We obtain

Theorem 2—*For* $\tau \geqq 8$

$$f(\tau) = 2 \sum_{\nu=1}^{m} \frac{\cos 2\pi(\kappa - \tau \log \nu)}{\nu^{\frac{1}{2}}} + g(\tau) \exp\left(-\frac{7i}{96\pi\tau} + \omega_6 + i\omega_7\right) + \mathrm{R}, \tag{6.1}$$

where

$$|\mathrm{R}| < \left(\frac{0\cdot4652}{1 - 0\cdot813\,\tau^{-\frac{1}{2}}} + \frac{0\cdot4168}{1 - 0\cdot489\,\tau^{-\frac{1}{2}}}\right)\frac{1}{\tau^{\frac{3}{4}}} + \frac{0\cdot969}{\tau^{\frac{5}{12}}}\,10^{-0\cdot4\tau^{\frac{1}{2}}}$$
$$+ \frac{0\cdot38}{\tau^{\frac{5}{12}}}\,10^{-0\cdot32\tau^{\frac{1}{2}}} + \frac{0\cdot309}{\tau^{\frac{3}{4}}}\,10^{-0\cdot45\tau} + \frac{0\cdot655}{\tau^{\frac{3}{4}}}\,10^{-2\cdot9\tau} + 0\cdot065\,10^{-3\tau}, \tag{6.2}$$

and

$$|\omega_6| < \frac{7}{160\pi^2\tau^2}, \quad |\omega_7| < \frac{1}{30\pi^2\tau^2}.$$

In the applications the factor multiplying $g(\tau)$ can be replaced by unity, with negligible error. The last three terms on the right-hand side of (6.2) are also quite negligible.

7—Let τ_n be the point where $\kappa = \frac{1}{2}n - 1$. Then the first term on the right-hand side of (6.1) is

$$2(-1)^n \sum_{\nu=1}^{m} \nu^{-\frac{1}{2}} \cos(2\pi\tau_n \log \nu),$$

and the sum begins with 1, and shows a general tendency to be positive. This suggests that $f(\tau_n)$ and $f(\tau_{n+1})$ will generally have opposite signs, and so that the interval (τ_n, τ_{n+1}) will contain a zero of $\zeta(\frac{1}{2} + 2\pi i\tau)$. This phenomenon is referred to by Hutchinson as Gram's law. It is known that the law is not universally valid, but it is true as far as the calculations go, with very few exceptions, and the verification of the Riemann hypothesis consists in the main of a verification of Gram's law.

The object of the calculations is to determine the sign of $f(\tau)$ at convenient points near to the points τ_n. For this purpose a table was constructed as follows. By Lemma ζ

$$\kappa = \tfrac{1}{2}(\tau \log \tau - \tau - \tfrac{1}{8}) + \frac{1}{192\pi^2\tau} + \frac{\omega_7}{2\pi}, \tag{7.1}$$

where $|\omega_7| < 1/(30\pi^2\tau^2)$. The last two terms may be neglected in the calculations. Values of τ are found which give κ approximately the values $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1, ... The first few values, $\tau = 1\cdot6$, $2\cdot84$, $3\cdot68$, ..., were found by trial, and then it soon becomes possible to write down sufficiently accurate values in succession by noticing the behaviour of the differences; τ has exactly the value given in the first column, and κ (which will not be exactly half an integer) is calculated to four decimal places. The result is given in the second column.

The values of $\alpha_\nu = \alpha_\nu(\tau) = \nu^{-\frac{1}{2}} \cos 2\pi(\kappa - \tau \log \nu)$, to three decimal places, are given in succeeding columns. To calculate the cosines, a table of $\cos 2\pi x$ was made giving four decimal places at intervals of $0\cdot0001$. This was made by finding the values to five decimal places at intervals of $0\cdot001$ from a standard table in degrees, etc., and then inserting the other values by linear interpolation, using a machine. This seems to be sufficiently accurate for the purpose in hand. The α_ν can be calculated quite quickly by using this table and a machine.

To calculate $g(\tau)$, let $\sqrt{\tau} = m + \xi$, where $0 \leqq \xi < 1$, and let $g(\tau) = (-1)^{m-1} \tau^{-\frac{1}{4}} h(\xi)$ (*see* 1.1). Clearly $h(\xi) = h(1 - \xi)$. A short table of $h(\xi)$ for $0 \leqq \xi \leqq \frac{1}{2}$ was made, showing that $h(\xi)$ decreases steadily from $0\cdot924$ to $0\cdot383$. In calculating $g(\tau)$ I also used Milne-Thomson's "Standard Table of Square Roots."

The main table was carried from $\tau = 1\cdot6$, where $\kappa = -0\cdot4865$, to $\tau = 62\cdot785$, where $\kappa = 98\cdot5010$; but it is only for $\tau > 8$ that the above analysis applies. Sufficiently small error terms could, no doubt, be obtained by this method for some of the values of τ less than 8, but it is not clear how to do this in all cases, and for some of the early values we should have to use another method.

τ	κ	α_1	α_2	α_3	α_4	α_5	α_6	$2\sum_{\nu=1}^{m}\alpha_\nu$	$g(\tau)$	Total
8·25	4·5172	−0·994	0·213					−1·562	−0·390	−1·952
8·71	5·0088	0·998	0·696					3·388	−0·473	2·915
9·16	5·5015	−1·000	0·407	−0·534				−2·254	0·495	−1·759
9·62	6·0166	0·995	−0·410	−0·547				0·076	0·402	0·478
10·05	6·5081	−0·999	−0·683	−0·565				−4·494	0·335	−4·159
16·28	14·5076	−0·999	0·119	−0·415	0·463			−1·664	−0·420	−2·084
16·63	14·9977	1·000	−0·695	−0·080	0·469			1·388	−0·373	1·015
16·98	15·4915	−0·999	−0·124	0·300	0·478			−0·690	−0·332	−1·022
17·34	16·0032	1·000	0·703	0·553	0·488			5·488	−0·296	5·192
17·69	16·5043	−1·000	0·033	0·523	0·496			0·104	−0·268	−0·164
17·73	*16·5617*	*−0·926*	*−0·098*	*0·500*	*0·497*			*−0·054*	*−0·265*	*−0·319*
18·04	17·0088	0·998	−0·707	0·213	0·500			2·008	−0·244	1·764
26·74	30·5035	−1·000	0·693	0·404	−0·458	−0·438		−1·598	0·262	−1·336
27·00	*30·9313*	*0·908*	*0·149*	*−0·068*	*−0·500*	*−0·442*		*0·094*	*0·247*	*0·341*
27·04	30·9972	1·000	−0·020	−0·144	−0·499	−0·443		−0·212	0·244	0·032
27·34	31·4927	−0·999	−0·682	−0·556	−0·420	−0·446		−6·206	0·229	−5·977
29·14	34·4992	−1·000	−0·222	−0·575	0·400	−0·361		−3·516	0·173	−3·343
29·44	35·0058	0·999	−0·573	−0·301	0·174	−0·318		−0·038	0·168	0·130
29·73	35·4969	−1·000	0·544	0·295	−0·101	−0·267		−1·058	0·166	−0·892
32·06	39·4933	−0·999	−0·093	−0·079	0·477	0·353		−0·682	0·181	−0·501
32·35	39·9967	1·000	−0·633	−0·556	0·294	0·406		1·022	0·188	1·210
32·64	40·5015	−1·000	0·507	−0·360	−0·009	0·439		−0·846	0·195	−0·651
32·92	40·9901	0·998	0·334	0·258	−0·302	0·447		3·470	0·204	3·674
44·69	62·4979	−1·000	−0·701	−0·469	−0·481	−0·402	−0·363	−6·832	−0·172	−7·004
44·95	62·9923	0·999	0·361	−0·446	−0·217	−0·267	−0·390	0·080	−0·177	−0·097
45·00	*63·0873*	*0·853*	*0·561*	*−0·340*	*−0·142*	*−0·234*	*−0·394*	*0·608*	*−0·180*	*0·428*
45·22	63·5064	−0·999	0·370	0·269	0·208	−0·063	−0·406	−1·242	−0·184	−1·426
45·48	64·0023	1·000	−0·700	0·562	0·479	0·152	−0·407	2·172	−0·190	1·982
45·74	64·4989	−1·000	0·194	0·006	0·422	0·332	−0·393	−0·878	−0·197	−1·075
46·00	64·9963	1·000	0·541	−0·559	0·073	0·435	−0·363	2·254	−0·205	2·049
46·26	65·4943	−0·999	−0·638	−0·270	−0·329	0·432	−0·319	−4·246	−0·214	−4·460
46·52	65·9932	0·999	−0·009	0·435	−0·500	0·322	−0·259	1·976	−0·224	1·752
46·78	66·4927	−0·999	0·645	0·468	−0·314	0·130	−0·187	−0·514	−0·234	−0·748
47·00	*66·9163*	*0·865*	*−0·373*	*−0·113*	*0·033*	*−0·064*	*−0·117*	*0·462*	*−0·243*	*0·219*
47·04	66·9930	0·999	−0·537	−0·227	0·099	−0·098	−0·105	0·262	−0·244	0·018
47·30	67·4940	−0·999	−0·184	−0·567	0·442	−0·301	−0·016	−3·250	−0·256	−0·3506

τ	κ	α_1	α_2	α_3	α_4	α_5	α_6	α_7	$g(\tau)$	Total
62·06	97·0025	1·000	0·704	0·254	0·491	0·324	0·140	0·025	0·240	6·116
62·30	97·4981	−1·000	−0·281	0·544	0·338	0·056	0·282	−0·042	0·249	0·043
62·35	*97·6014*	*−0·804*	*−0·527*	*0·461*	*0·252*	*−0·008*	*0·306*	*−0·047*	*0·251*	*−0·483*
62·545	98·0045	1·000	−0·410	−0·150	−0·151	−0·245	0·378	−0·111	0·258	0·880
62·785	98·5010	−1·000	0·703	−0·571	−0·486	−0·427	0·408	−0·176	0·269	−2·829

8—I give a few specimens of the table.

The upper bound for the remainder is not tabulated, since it is steadily decreasing, and to determine the sign of $f(\tau)$ it is only necessary to calculate it for occasional values of τ.

The values $\tau = 8 \cdot 25$ to $10 \cdot 05$ are the first five to which Theorem 2 applies. The total $0 \cdot 478$ for $\tau = 9 \cdot 62$ is a good deal the smallest in absolute value in this part of the table. I find $0 \cdot 33$ as an upper bound for the remainder at this point, so that $f(\tau)$ is certainly positive.

The first doubtful point is $\tau = 17 \cdot 69$. Here the value obtained for $\Sigma \alpha_\nu$ has the "wrong" sign. The value obtained for the total is negative, but is smaller in absolute value than the upper bound for the remainder, viz., $0 \cdot 182$. The sign of $f(\tau)$ is therefore not determined. An additional point $17 \cdot 73$ is therefore considered, and here $f(\tau)$ is certainly negative. Hence there are two zeros between $\tau = 17 \cdot 34$ and $\tau = 18 \cdot 04$.

A similar state of affairs is found at $\tau = 27 \cdot 04$, and at $29 \cdot 44$ the remainder is less than $0 \cdot 113$.

The point $\tau = 32 \cdot 06$ corresponds to the first entry in the table on p. 60 of Hutchinson's paper. The value of $C(\gamma_\nu)$ given by Hutchinson corresponds to $(-1)^k$ times the "total" obtained here, where k is the nearest integer to 2κ. In every case there is a reasonably good agreement between the results.

At $\tau = 44 \cdot 59$, $f(\tau)$ actually has the "wrong" sign, the remainder here being less than $0 \cdot 08$. At $\tau = 45 \cdot 00$, however, $f(\tau)$ is positive, so that the expected number of zeros is found. This is the first point noticed by Hutchinson at which "Gram's law" fails.

The next point which he notices is $\tau = 47 \cdot 04$; but here the present method fails to determine the sign of $f(\tau)$. However, $f(47 \cdot 00)$ is positive, so that the expected zeros are found.

The remainder of the table is chiefly remarkable for its regularity. The only other point where $f(\tau)$ may possibly have the wrong sign is $\tau = 62 \cdot 3$; and $f(62 \cdot 35)$ is negative, so that the zeros are found as usual.

Most of the calculations have not been verified, but the agreement with Hutchinson seems to suggest that they are fairly accurate. Actually the behaviour of $\alpha_\nu(\tau)$, for a fixed ν, as τ increases, makes it easy to detect a gross error. The term α_ν first appears when $\tau = \nu^2$, and here, by (7.1),

$$\cos 2\pi(\kappa - \tau \log \nu) = \cos \pi \{\tau \log(\tau/\nu^2) - \tau - \tfrac{1}{8} + \ldots\}$$
$$= \cos \pi(\nu^2 + \tfrac{1}{8} + \ldots) = (-1)^\nu \cos \tfrac{1}{8}\pi + \ldots,$$

and

$$\frac{d}{d\tau}(\kappa - \tau \log \nu) = \tfrac{1}{2} \log \tau - \log \nu - \frac{1}{192\pi^2\tau^2} + \ldots = -\frac{1}{192\pi^2\tau^2} + \ldots .$$

At its first appearance in the table, α_ν will therefore be approximately $(-1)^\nu \nu^{-\frac{1}{2}} \cos \frac{1}{8}\pi$, and it may be expected to vary slowly for some time; see, for example, the values of α_4 beginning at $\tau = 16 \cdot 28$.

9—Our main argument gives a lower bound for the number of zeros of $\zeta(s)$ on $\sigma = \frac{1}{2}$ in certain intervals of values of t. We next obtain an upper bound for the whole number of zeros of $\zeta(s)$ for $0 < t \leqq T$, for certain values of T.

It is known that, if N (T) denotes this number,

$$\pi N(T) = \Delta \{\text{am}\, s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s)\},$$

where Δ denotes the variation from 2 to $2 + iT$, and thence to $\frac{1}{2} + iT$, along straight lines. Now

$$\Delta\, \text{am}\, s(s-1) = \pi. \quad \Delta\, \text{am}\, \pi^{-\frac{1}{2}s} = -\tfrac{1}{2} T \log \pi,$$

and

$$\Delta\, \text{am}\, \Gamma(\tfrac{1}{2}s) = \mathbf{I} \log \Gamma(\tfrac{1}{4} + \tfrac{1}{2}iT) = 2\pi\kappa(T) + \tfrac{1}{2}T \log \pi.$$

Putting $\text{am}\, \zeta(\frac{1}{2} + it) = S(t)$, we therefore have

$$N(T) = 2\kappa(T) + 1 + S(T)/\pi.$$

Now

$$\text{am}\, \zeta(s) = \text{arc tan}\, \{\mathbf{I}\zeta(s)/\mathbf{R}\zeta(s)\}.$$

We know that $\mathbf{R}\zeta(s)$ does not vanish along $\sigma = 2$. If T is such that $\mathbf{R}\zeta(s)$ does not vanish for $\frac{1}{2} \leqq \sigma \leqq 2$, $t = T$, it follows that $|S(\tau)| < \frac{1}{2}\pi$. Hence, if k is the integer nearest to $2\kappa(T)$,

$$|N(T) - k - 1| < 1,$$

and so, N (T) being an integer,

$$N(T) = k + 1.$$

It is therefore a question of approximating to $\mathbf{R}\zeta(s)$ for $\frac{1}{2} \leqq \sigma \leqq 2$. We write

$$\zeta(s) = \sum_{\nu=1}^{m} \frac{1}{\nu^s} + \left(\frac{2\pi}{t}\right)^{\sigma-\frac{1}{2}} \chi(\tfrac{1}{2} + it) \sum_{=1}^{m} \frac{1}{\nu^{1-s}}$$
$$+ \left\{\chi(s) - \left(\frac{2\pi}{t}\right)^{\sigma-\frac{1}{2}} \chi(\tfrac{1}{2} + it)\right\} \sum_{\nu=1}^{m} \frac{1}{\nu^{1-s}} - \frac{\Gamma(1-s)}{2\pi i} J(\sigma),$$

and, taking real parts,

$$\mathbf{R}\zeta(s) = \sum_{\nu=1}^{m} \left\{\frac{\cos 2\pi\tau \log \nu}{\nu^\sigma} + \frac{\cos 2\pi(2\kappa - \tau \log \nu)}{\tau^{\sigma-\frac{1}{2}}\nu^{1-\sigma}}\right\} + M_1 + M_2,$$

where

$$|M_1| \leq \left|\chi(s) - \left(\frac{2\pi}{t}\right)^{\sigma-\frac{1}{2}} \chi(\tfrac{1}{2}+it)\right| \sum_{\nu=1}^{m} \frac{1}{\nu^{1-\sigma}}$$

$$< \left|\chi(s) - \left(\frac{2\pi}{t}\right)^{\sigma-\frac{1}{2}} \chi(\tfrac{1}{2}+it)\right| \left(\frac{m^\sigma}{\sigma} + m^{\sigma-1}\right),$$

and

$$|M_2| < \frac{|\Gamma(1-s)|}{2\pi} |J(\sigma)|.$$

We shall not pursue these calculations in detail, but it is easy enough to see that $\mathbf{R}\,\zeta(s) > 0$ $(\frac{1}{2} \leq \sigma \leq 2)$ for certain values of t. Take, for example, $\tau = 62{\cdot}06$ (suggested by the table). The cosines in the above formula are all positive, so that

$$\mathbf{R}\,\zeta(s) > 1 - |M_1| - |M_2|.$$

For large t the upper bound for $|M_1|$ is approximately

$$(\sigma - \tfrac{1}{2})^2 (2\sigma)^{-1} \tau^{-\frac{1}{2}\sigma-\frac{1}{4}}.$$

Theorem 1, combined with lemma ε, gives an upper bound for $|M_2|$. Apart from a few minor terms, it is at its greatest when $\sigma = \frac{1}{2}$, the case already considered.

We therefore conclude that

$$N(2\pi \times 62{\cdot}06) = 195.$$

This, however, is the lower bound obtained† for the number of real zeros of $f(\tau)$ for $0 < \tau < 62{\cdot}06$. Hence $\zeta(s)$ has 195 zeros on $\sigma = \frac{1}{2}$ between $t = 0$ and $t = 2\pi \times 62{\cdot}06$, and no other zero between these values of t.

10—So far as the above calculations go, the method of separating the zeros of $\zeta(\frac{1}{2}+it)$ by means of the points $t_n = 2\pi\tau_n$ is almost completely successful. We find that $(-1)^n f(\tau_n)$ is positive for all values of n up to $n = 199$ with at most three exceptions, and that in these cases $\zeta(\frac{1}{2}+it)$ has a zero only just outside the usual interval (t_n, t_{n+1}).

Nevertheless, this process cannot go on in the same way indefinitely. We shall now show not merely that $(-1)^n f(\tau_n)$ cannot be positive for all values of n (which the calculations have already shown), but that it cannot be positive for all sufficiently large values of n.

The proof consists mainly of a combination of known theorems on $\zeta(s)$.

† We rely on previous calculations for the first few zeros.

Let $N(T)$ be the number of zeros $\beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma \leq T$. It is known that

$$N(T) = M(T) + R(T), \tag{10.1}$$

where

$$M(T) = \frac{1}{2\pi} T \log T - \frac{1 + \log 2\pi}{2\pi} T + \tfrac{7}{8}, \tag{10.2}$$

and

$$\int_0^T R(t)\,dt = O(\log T). \tag{10.3}$$

It follows that $R(t) = 0$ *for arbitrarily large values of* t. For let the zeros of $\zeta(s)$ with positive imaginary parts be arranged in order $\beta_n + i\gamma_n$ of non-decreasing ordinates (multiple zeros occurring with the right order of multiplicity). Suppose that $\gamma_n < \gamma_{n+1}$ for a certain value of n. Then $N(t) = n$ for $\gamma_n \leq t < \gamma_{n+1}$. Since $M(t)$ is steadily increasing, $R(t)$ is steadily decreasing in this interval. Let $l(t)$ be the linear function of t which takes the values $R(\gamma_n)$ and $R(\gamma_{n+1} - 0)$ at γ_n and γ_{n+1} respectively. Then in this interval

$$l(t) - R(t) = \{R(\gamma_{n+1} - 0) - R(\gamma_n)\} \frac{t - \gamma_n}{\gamma_{n+1} - \gamma_n} - \{R(t) - R(\gamma_n)\}$$

$$= -\{M(\gamma_{n+1}) - M(\gamma_n)\} \frac{t - \gamma_n}{\gamma_{n+1} - \gamma_n} + \{M(t) - M(\gamma_n)\}$$

$$= -M'(\xi_1)(t - \gamma_n) + M'(\xi_2)(t - \gamma_n)$$

$$= M''(\xi)(\xi_2 - \xi_1)(t - \gamma_n),$$

where $\gamma_n < \xi_1 < \gamma_{n+1}$, $\gamma_n < \xi_2 < t$, and ξ lies between ξ_1 and ξ_2. Now

$$M''(t) = \frac{1}{2\pi t}.$$

Hence

$$l(t) - R(t) = O\{(\gamma_{n+1} - \gamma_n)^2/\gamma_n\} = O(1/\gamma_n),$$

since $\gamma_{n+1} - \gamma_n$ is bounded.

Suppose now that $R(t) \geq 0$ for $t > a$. Since there is at least one zero at γ_n,

$$N(\gamma_n) - N(\gamma_n - 0) \geq 1.$$

Since $M(t)$ is continuous, it follows that

$$R(\gamma_n) - R(\gamma_n - 0) \geq 1,$$

and so, on our hypothesis, $R(\gamma_n) \geqq 1$. Hence

$$\int_{\gamma_n}^{\gamma_{n+1}} R(t)\,dt = \int_{\gamma_n}^{\gamma_{n+1}} l(t)\,dt + O\{(\gamma_{n+1} - \gamma_n)/\gamma_n\}$$
$$= \tfrac{1}{2}(\gamma_{n+1} - \gamma_n)\{R(\gamma_n) + R(\gamma_{n+1} - 0)\} + O\{(\gamma_{n+1} - \gamma_n)/\gamma_n\}$$
$$\geqq \tfrac{1}{2}(\gamma_{n+1} - \gamma_n) + O\{(\gamma_{n+1} - \gamma_n)/\gamma_n\}$$
$$\geqq \tfrac{1}{4}(\gamma_{n+1} - \gamma_n)$$

for sufficiently large n. If $\gamma_{n+1} = \gamma_n$, both sides of the inequality are 0. Hence, summing with respect to n from a sufficiently large starting-point n_0,

$$\int_{\gamma_{n_0}}^{\gamma_N} R(t)\,dt \geqq \tfrac{1}{4}(\gamma_N - \gamma_{n_0}),$$

which contradicts (10.3). A similar contradiction is obtained if $R(t) \leqq 0$ for $t > a$.

Since $R(t)$ can only pass from a positive to a negative value by passing through 0, it follows that $R(t) = 0$ for some arbitrarily large values of t.

Suppose now that $(-1)^n f(t_n) > 0$ for $n \leqq n_1$. Then if $t_n < t < t_{n+1}$

$$N_0(t) \geqq n - n_1 = 2\kappa(t_{n+1}) - n_1 > 2\kappa(t) - n_1 > M(t) - O(1),$$

and so

$$N_0(t) > M(t) - O(1)$$

for all values of t. Let t^* denote a sequence tending to infinity such that $R(t^*) = 0$. Then $N(T^*) = M(T^*)$. Hence

$$N_0(t^*) > N(T^*) - O(1).$$

Hence the number of complex zeros of $\zeta(s)$ not on the line $\sigma = \frac{1}{2}$ is finite. Hence for all values of t

$$N(t) = N_0(t) + O(1) \geqq M(t) + O(1),$$

i.e.,

$$R(t) \geqq O(1).$$

But it is known† that, on the Riemann hypothesis,

$$R(t) < -(\log t)^c$$

for arbitrarily large values of t, c being an absolute constant; and the possible existence of a finite number of complex zeros of $\zeta(s)$ not on $\sigma = \frac{1}{2}$ does not invalidate this result. The hypothesis that $(-1)^\nu f(t_\nu)$ is ultimately positive is therefore untenable.

† Bohr and Landau, 'Math. Ann.,' vol. 74, p. 3 (1913).

It can be proved in the same way that, if c_n is the nth zero of $\zeta(\frac{1}{2} + it)$, then $(c_n - t_n)/(t_{n+1} - t_n)$ is unbounded; for if it is bounded we again deduce that $N_0(t) - M(t)$ is bounded.

11—The occurrence of so many zeros of $f(s)$ on $\sigma = \frac{1}{2}$ might be claimed as an argument in favour of the Riemann hypothesis, but I think that a closer examination of the situation shows that it is not a very strong argument. The calculations show that departures from "Gram's law," that the numbers c_n and t_n interlace, are very slow in occurring; and yet arbitrarily wide departures from this law, in the sense of the remark at the end of the last section, must ultimately occur. A still wider departure from regularity such as would be needed to produce a zero off $\sigma = \frac{1}{2}$ might well be still more remote.

There are relations between the numbers $\cos(t \log \nu)$ which have a certain influence in making $(-1)^n f(t_n)$ positive. For example, if θ, ϕ, and ψ are independent,

$$\min\left(1 + \frac{\cos\theta}{\sqrt{2}} + \frac{\cos\phi}{\sqrt{3}} + \frac{\cos\psi}{\sqrt{6}}\right) = 1 - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}} < 0;$$

but

$$\min\left(1 + \frac{\cos\theta}{\sqrt{2}} + \frac{\cos\phi}{\sqrt{3}} + \frac{\cos(\theta + \phi)}{\sqrt{6}}\right) = \tfrac{1}{12},$$

so that

$$1 + \frac{\cos(t \log 2)}{\sqrt{2}} + \frac{\cos(t \log 3)}{\sqrt{3}} + \frac{\cos(t \log 6)}{\sqrt{6}} \geqq \quad .$$

Apparently such relations succeed in preserving "Gram's law" so long as the series $\sum_{\nu \leqq \sqrt{\tau}} \nu^{-\frac{1}{2}} \cos(t \log \nu)$ does not contain too many terms, but exceptions ultimately occur. The calculations do not suggest any reason why exceptions to "Riemann's law," that the zeros lie on $\sigma = \frac{1}{2}$, should not ultimately occur too.

Summary

An approximate formula for the Riemann zeta-function in the critical strip is obtained in a form suitable for calculation. It is used to show that the first 195 zeros in the upper half-plane lie on the critical line. The problem of the zeros is also investigated theoretically.