Apart from the δ-functions, one often uses other improper functions, such as, for instance,

$$\delta_+(x) = \delta_-^*(x) = \frac{1}{2\pi i} \lim_{\alpha \to 0} \frac{1}{x - i\alpha}. \tag{A 20}$$

Using (A 20) and (A 6) we find

$$\left.\begin{aligned} \delta_+(x) + \delta_-(x) &= \lim_{\alpha \to 0} \frac{1}{\pi} \frac{\alpha}{x^2 + \alpha^2} = \delta(x), \\ \delta_+(x) - \delta_-(x) &= \lim_{\alpha \to 0} \frac{1}{\pi i} \frac{x}{x^2 + \alpha^2}. \end{aligned}\right\} \tag{A 21}$$

The delta-function $\delta(z)$ as a function of a complex variable has two simple poles at the points ix and $-ix$ with residues equal to $1/2\pi i$ and $-1/2\pi i$, respectively. When integrating expressions containing $\delta(z)$, the integration path must go between these poles. Equation (A 20) and (A 21) remain valid also for complex values of x. In that case, we have

$$\delta_-(z) = \delta_+(-z) = \delta_+^*(z) = [\delta_+(z^*)]^*.$$

The functions $\delta_+(z)$ and $\delta_-(z)$ can be written in the form

$$\delta_+(z) = \frac{1}{2\pi i z}, \quad \delta_-(z) = -\frac{1}{2\pi i z},$$

if we remember to take the path of integration above and below the point $z = 0$, respectively.

B. The Angular Momentum Operators in Spherical Coordinates

We gave in Section 7 expressions for the components of the angular momentum operator in Cartesian coordinates:

$$\hat{L}_z = -i\hbar \left[x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right], \ldots \tag{B 1}$$

We shall now find the form of these operators in spherical polars. The transformation

$$x = r \sin\theta \cos\varphi, \quad y = r \sin\theta \sin\varphi, \quad z = r\cos\theta,$$

has its inverse

$$r^2 = x^2 + y^2 + z^2, \quad \cos\theta = \frac{z}{r}, \quad \tan\varphi = \frac{y}{x}.$$

Hence we have

$$\frac{\partial r}{\partial z} = \cos\theta, \quad \frac{\partial r}{\partial y} = \sin\theta \sin\varphi, \quad \frac{\partial r}{\partial x} = \sin\theta\cos\varphi,$$

$$\frac{\partial \theta}{\partial z} = -\frac{\sin\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\cos\theta \sin\varphi}{r}, \quad \frac{\partial \theta}{\partial x} = \frac{\cos\theta\cos\varphi}{r},$$

$$\frac{\partial \varphi}{\partial z} = 0, \quad \frac{\partial \varphi}{\partial y} = \frac{\cos\varphi}{r\sin\theta}, \quad \frac{\partial \varphi}{\partial x} = -\frac{\sin\varphi}{r\sin\theta}.$$

Using these equations we find

$$\hat{L}_z = -i\hbar\left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right]$$

$$= -i\hbar\left\{r\sin\theta\cos\varphi\left[\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} + \frac{\partial\varphi}{\partial y}\frac{\partial}{\partial\varphi}\right]\right.$$

$$\left. - r\sin\theta\sin\varphi\left[\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi}\right]\right\}$$

$$= -i\hbar\frac{\partial}{\partial\varphi}. \tag{B 2}$$

Similarly, we find

$$\hat{L}_x = i\hbar\left[\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right], \tag{B 3}$$

$$\hat{L}_y = -i\hbar\left[\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right], \tag{B 4}$$

and thus

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]. \tag{B 5}$$

One often uses, instead of the operators $\hat{L}_x$ and $\hat{L}_y$, the linear combinations

$$\hat{L}_+ \equiv \hat{L}_x + i\hat{L}_y = \hbar e^{i\varphi}\left[\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right],$$

$$\hat{L}_- \equiv \hat{L}_x - i\hat{L}_y = \hbar e^{-i\varphi}\left[-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right].$$

C. Linear Operators in a Vector Space; Matrices

To make it easier to use this book, we remind the reader of some definitions connected with vector space of a finite or an infinite dimensionality. The concept of a vector space is a generalisation of the concept of the normal three-dimensional space.

I. The infinite set of complex quantities $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$ for which the linear operations of addition and multiplication by complex numbers are defined is called a complex *vector space* $\boldsymbol{R}$. The quantities $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$ themselves are called the vectors of the space $\boldsymbol{R}$.

The vector space $\boldsymbol{R}$ is a linear space, that is, it has the property that any linear combination of two vectors—such as $a\boldsymbol{A} + b\boldsymbol{B}$ where a and b are complex numbers—forms a vector belonging to the same vector space. To each pair of vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ in the vector space, we can assign a number $\langle \boldsymbol{A}|\boldsymbol{B}\rangle$ or $(\boldsymbol{A}\cdot\boldsymbol{B})$, the so-called scalar product of vectors. The definition of the scalar product will be given in subsection IV of this section.