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Systematic review of the generalized Bell inequalities*

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ABSTRACT

In the systematic review of the Bell inequalities, the system of inequalities (34) was given in a more precise way and the new inequality of the Bell type was also constructed. The suggestion of an experimental test of this new inequality is proposed.

1. INTRODUCTION

The problem began with the Einstein Podolsky Rosen paradox [1] which in a modification given by Bohr and Aharonov [2] is the following. Consider two electrons in the initial singlet state ($S = 0$) freely moving in opposite directions. The spin of each electron is then measured after some time of a separation in remote places A and B (Fig. 1). If a measured spin of an electron in A is in a direction $\boldsymbol{a}$ (say: spin up) then, due to a quantum mechanical prediction, a spin of an electron in B must be in a direction $-\boldsymbol{a}$ (say: spin down). However, in classical physics, the measurement on the particle in A does not influence the particle in B. In what follows, the spin will be taken in the unit $\hbar/2$ and an eigenvalue of a spin component will be $\sigma_k = \pm 1$.

Let us generalize the problem and ask the following question. What is an expectation value of the electron spin in B measured in the direction $\boldsymbol{b}$ when the spin of the electron in A has been measured in the direction $\boldsymbol{a}$ (Fig. 1)? It is clear that an answer will be different in classical and in quantum physics,

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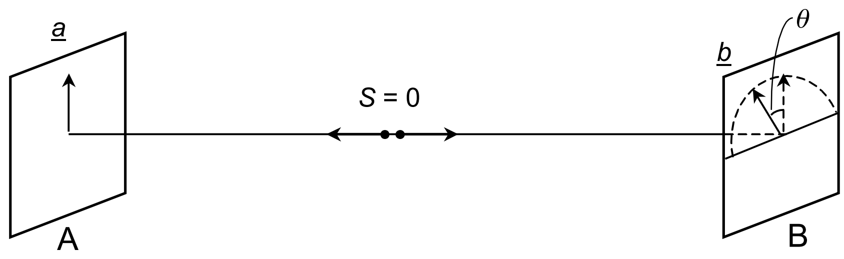


Fig. 1. Einstein Podolsky Rosen paradox with electron spins

because the first does not predict any correlations of two electrons while the second necessarily imposes a strong spin-correlations. Results of a single measurement in B can give only $+1$ (the spin in a direction $\boldsymbol{b}$) or -1 (the spin in $-\boldsymbol{b}$). Hence a mean value of spin measurements $P(ab)$ is bounded

$$-1 \leq P(ab) \leq 1\,.$$

According to quantum mechanics, calculations give

$$P(ab) = -\cos\theta\,. \tag{1}$$

The fundamental result of Bell work [3] gave

$$P(ab)_{\text{class}} \neq -\cos\theta\,. \tag{2}$$

It is the conclusion followed the Bell inequality for any classical $P(ab)$

$$1 + P(bb') \geq |P(ab) - P(ab')| \tag{3}$$

Modifications and generalizations of the Bell inequality have been directed toward forms easily applied to experiments both with electrons and with photons initially coupled to $S = 0$. Quantum calculations gave for photons

$$P(ab) = \cos 2\theta\,. \tag{4}$$

The aim of the present work is the systematic comparison of the different Bell type inequalities including the new one obtained here. In the last Section the particular classical model is introduced which gives the same, as in quantum mechanics, predictions for special angles between $\boldsymbol{a}$ and $\boldsymbol{b}$.

2. QUANTUM MECHANICAL CORRELATIONS OF THE EPR PARADOX WITH ELECTRONS

Similarly, as in the introduction, we will consider the correlations of electron spins in A (direction $\boldsymbol{a}$) and in B (direction $\boldsymbol{b}$).

After a spin measurement of the electron in A in the $\boldsymbol{a}$ direction, the second electron in B must have a spin in the direction $-\boldsymbol{a}$, but measurement is performed in the direction $\boldsymbol{b}$. We need to calculate the probability of an electron jump from $-\boldsymbol{a}$ to $\boldsymbol{b}$. The vector state $|b\rangle$ can be formed in the orthogonal base $|\boldsymbol{a}\rangle$ and $|-\boldsymbol{a}\rangle|$

$$|\boldsymbol{b}\rangle = k_1|\boldsymbol{a}\rangle + k_2|-\boldsymbol{a}\rangle$$

or

$$\frac{1}{k_1}|\boldsymbol{b}\rangle \equiv |\boldsymbol{b}\rangle' = |\boldsymbol{a}\rangle + q|-\boldsymbol{a}\rangle \tag{5}$$

where k_1 and k_2 are probability amplitudes. From a projection construction of the Riemann sphere on a plane [9] we get

$$|q| = \tan\frac{\theta}{2} \qquad\qquad 0 \le \theta \le \pi\,. \tag{6}$$

Then

$$|\boldsymbol{b}\rangle' = |\boldsymbol{a}\rangle + \tan\frac{\theta}{2}|-\boldsymbol{a}\rangle\,. \tag{7}$$

After normalization

$$|\boldsymbol{b}\rangle = \cos\frac{\theta}{2}|\boldsymbol{a}\rangle + \sin\frac{\theta}{2}|-\boldsymbol{a}\rangle \tag{8}$$

Similarly

$$|-\boldsymbol{b}\rangle' = |\boldsymbol{a}\rangle + q'|-\boldsymbol{a}\rangle\,, \tag{9}$$

where $q' = \cot\theta/2$.
Hence

$$|-\boldsymbol{b}\rangle' = \sin\frac{\theta}{2}|\boldsymbol{a}\rangle + \cos\frac{\theta}{2}|-\boldsymbol{a}\rangle\,. \tag{10}$$

Immediately we get

$$|\langle -\boldsymbol{a}|\boldsymbol{b}\rangle|^2 = \sin^2\frac{\theta}{2} = \frac{1}{2}(1-\cos\theta) \tag{11}$$

$$|\langle -\boldsymbol{a}|-\boldsymbol{b}\rangle|^2 = \cos^2\frac{\theta}{2} = \frac{1}{2}(1+\cos\theta)\,. \tag{12}$$

Hence, the expectation value of the spin measurement in B ($\boldsymbol{b}$) is

$$P(ab) = |\langle -\boldsymbol{a}|\boldsymbol{b}\rangle|^2 \times (+1) + |\langle -\boldsymbol{a}| - \boldsymbol{b}\rangle|^2 \times (-1) = -\cos\theta\,. \qquad (13)$$

We introduce now the notation "yes" (+1) and "no" (−1) for spins "up" and "down". Hence, the spin of the state $|\boldsymbol{b}\rangle$ will be marked by "yes" and $|-\boldsymbol{b}\rangle$ by "no"; contrary, the spin of $|-\boldsymbol{a}\rangle$ — by "yes" and $|\boldsymbol{a}\rangle$ by "no". Probabilities of the four possible correlations read (where we have taken the factor 1/2 because we have introduced two possibilities $\boldsymbol{a}$ and $-\boldsymbol{a}$)

$$\begin{array}{ll}
\text{"yes yes"} & w_{++} \equiv \tfrac{1}{2}|\langle -\boldsymbol{a}|\boldsymbol{b}\rangle|^2 = \tfrac{1}{2}\sin^2\tfrac{\theta}{2} \\
\text{"no no"} & w_{--} \equiv \tfrac{1}{2}|\langle \boldsymbol{a}|-\boldsymbol{b}\rangle = \tfrac{1}{2}\sin^2\tfrac{\theta}{2} \\
\text{"yes no"} & w_{+-} \equiv \tfrac{1}{2}|\langle -\boldsymbol{a}|-\boldsymbol{b}\rangle = \tfrac{1}{2}\cos^2\tfrac{\theta}{2} \\
\text{"no yes"} & w_{-+} \equiv \tfrac{1}{2}|\langle \boldsymbol{a}|\boldsymbol{b}\rangle = \tfrac{1}{2}\cos^2\tfrac{\theta}{2}
\end{array} \qquad (14)$$

Hence

$$w_{++} + w_{--} + w_{+-} + w_{-+} = 1 \qquad (15)$$

and

$$\begin{aligned}
w_{++} = w_{--} &= \frac{1}{2}\sin^2\frac{\theta}{2} \\
w_{+-} = w_{-+} &= \frac{1}{2}\cos^2\frac{\theta}{2}\,.
\end{aligned} \qquad (16)$$

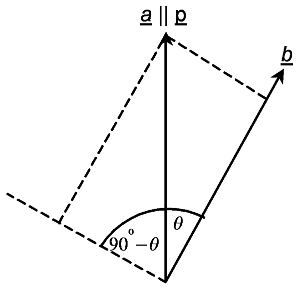
The expectation value $P(ab)$ can also be obtained from the formula

$$P(ab) = w_{++} + w_{--} - w_{+-} - w_{-+} = -\cos\theta \qquad (17)$$

3. QUANTUM MECHANICAL CORRELATIONS OF THE EPR PARADOX WITH PHOTONS

For a photon spin ($S = 1$) an abstract spin space is of three dimensions which makes a treatment more simple and transparent then for electrons. Suppose two photons in the initial single state ($S = 0$) are moving in opposite directions toward two polarisers in $A(\boldsymbol{a})$ and $B(\boldsymbol{b})$ with the angle θ between $\boldsymbol{a}$ and $\boldsymbol{b}$. Experimental results can be also described in "yes" and "no" notation with, however, different interpretation: "yes" (+) — for a parallel photon polarisation (the photon passes the polariser) and "no" − for a perpendicular polarisation (the photon is captured by the polariser). We will consider four cases "yes" "no" like in (14) using a simple geometrical interpretation.

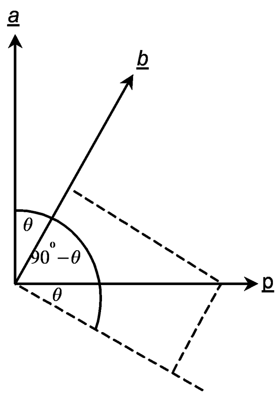
(i) The first photon is detected by the $\boldsymbol{a}$ polariser ($\boldsymbol{p} \parallel \boldsymbol{a}$) (and then, the second photon has the same polarisation direction).



Hence

$$\omega'_{++} = \cos^2\theta$$
$$w'_{+-} = \cos^2(90^\circ - \theta) = \sin^2\theta\,.$$

(ii) The first photon is captured by the $\boldsymbol{a}$ polariser ($\boldsymbol{p} \perp \boldsymbol{a}$).



Hence

$$\omega'_{-+} = \cos^2(90^\circ - \theta) = \sin^2\theta$$
$$w'_{--} = \cos^2\theta\,.$$

Hence

$$w'_{++} + w'_{--} + w'_{+-} + w'_{-+} = 2\cos^2\theta + 2\sin^2\theta = 2\,.$$

After normalization

$$w_{++} = w_{--} = \frac{1}{2}\cos^2\theta$$
$$w_{+-} = w_{-+} = \frac{1}{2}\sin^2\theta$$

and

$$P(ab) = w_{++} + w_{--} - w_{+-} - w_{-+} \equiv \cos^2\theta - \sin^2\theta = \cos 2\theta\,. \qquad (18)$$

Let us notice the characteristic difference of the correlation function $P(ab)$ for electrons ($-\cos\theta$) and for photons ($\cos 2\theta$).

4. CLASSICAL APPROACH TO THE EPR PARADOX

We assume, after Bell [3], that a complete specification of experimental results can be done by means of a hidden parameter λ. Spin (polarisation) measurements in A and B are

$$A(a\lambda) = \pm 1 \qquad \text{and} \qquad B(b\lambda) = \pm 1 \,.$$

We also assume that a measurement in B does not depend on the result in A. Similarly, we introduce the probability distribution $\rho(\lambda)$ with

$$\int d\lambda \rho(\lambda) = 1 \,.$$

The expectation values of $\boldsymbol{\sigma}_1$ in A and simultaneously $\boldsymbol{\sigma}_2$ in B is

$$P(ab) = \int d\lambda \rho(\lambda)\, A(a\lambda) B(b\lambda) \,.$$

We introduce also the correlation function $P(ab')$ for measurements in $\boldsymbol{a}$ and $\boldsymbol{b}'$. Then

$$P(ab) - P(ab') = \int d\lambda \rho(\lambda) \{A(a\lambda)B(b\lambda) - A(a\lambda)B(b'\lambda)\} \,.$$

Hence

$$|P(ab) - P(ab')| \le \int d\lambda \rho(\lambda) |A(a\lambda)B(b\lambda)| \{1 - B(b\lambda)B(b'\lambda)\} \,,$$

or

$$|P(ab) - P(ab')| \le 1 - \int d\lambda \rho(\lambda)\, B(b\lambda)B(b'\lambda) \,. \tag{19}$$

To calculate the integral in (19) we will follow Clauser et al. [4] procedure after some modifications. At first, we introduce the correlation function $P(a'b)$

$$\begin{aligned} P(a'b) &= -1 && \text{for } \boldsymbol{a}' = \boldsymbol{b} \\ P(a'b) &= -1 + \delta && \text{for } \boldsymbol{a}' \ne \boldsymbol{b} \qquad 0 < \delta \le 2 \,. \end{aligned} \tag{20}$$

In the integral

$$P(a'b) = \int \rho(\lambda) d\lambda\, A(a'\lambda)\, B(b\lambda)$$

we divide the λ-space on two regions

$$\Gamma_+ \qquad \text{for} \qquad A(a'\lambda) = B(b\lambda)$$

and

$$\Gamma_- \qquad \text{for} \qquad A(a'\lambda) = -B(b\lambda)\,. \tag{21}$$

From (20) and (21) we get

$$\int\limits_{\Gamma_+} d\lambda\rho(\lambda) = \frac{\delta}{2}\,; \qquad \int\limits_{\Gamma_-} d\lambda\rho(\lambda) = 1 - \frac{\delta}{2} \tag{22}$$

and also

$$\int d\lambda\rho(\lambda)B(b\lambda)B(b'\lambda) = \int d\lambda\rho(\lambda)A(a'\lambda)B(b'\lambda) - 2\int\limits_{\Gamma_-} d\lambda\rho(\lambda)A(a'\lambda)B(b'\lambda)$$

Hence

$$\int d\lambda\rho(\lambda)B(b\lambda)B(b'\lambda) \geq P(a'b') - 2\int_{\Gamma_-} d\lambda\rho(\lambda)$$

and by (22)

$$\int d\lambda\rho(\lambda)\, B(b\lambda)\, B(b'\lambda) \geq P(a'b') - 2 + \delta\,. \tag{23}$$

Taking (23) in (19), we get

$$|P(ab) - P(ab')| \leq 1 - P(a'b') + 2 - \delta$$

or

$$|P(ab) - P(ab')| \leq 2 - P(a'b') - P(a'b)\,. \tag{24}$$

Hence

$$|P(ab) - P(ab')| + P(a'b) + P(a'b') \leq 2\,. \tag{25}$$

The last inequality was obtained by Clauser et al. [4]. If we specify in (25) $P(a'b) = 1$ for $\boldsymbol{a}' = -\boldsymbol{b}$, we get

$$|P(ab) - P(ab')| \leq 1 + P(bb') \tag{3}$$

and this is exactly the Bell inequality (3).

Now we will perform an alternative procedure to calculate the integral in (19):

$$\int d\lambda\rho(\lambda)B(b\lambda)B(b'\lambda) = 2\int\limits_{\Gamma_+} d\lambda\rho(\lambda)A(a'\lambda)B(b'\lambda) - \int d\lambda\rho(\lambda)A(a'\lambda)B(b'\lambda)\,.$$

Then

$$\int d\lambda\rho(\lambda)B(b\lambda)B(b'\lambda) \geq -2\int\limits_{\Gamma_+} d\lambda\rho(\lambda) - P(a'b')\,.$$

or

$$\int d\lambda\rho(\lambda)B(b\lambda)B(b'\lambda) \geq -\delta - P(a'b')\,. \tag{26}$$

Taking (26) to (19) we get

$$|P(ab) - P(ab')| \leq 1 + \delta + P(a'b')$$

or

$$|P(ab) - P(ab')| \leq 2 + P(a'b') + P(a'b)\,. \tag{27}$$

One could not say whether the inequality (24) or (27) provides the better bounding for $|P(ab) - P(ab')|$, because for any $P: \quad -1 \leq P \leq 1$.

Let us combine both (24) and (27). From (24) we get

$$-2 + P(a'b') + P(a'b) \leq P(ab) - P(ab') \leq 2 - P(a'b') - P(a'b)$$

or

$$-2 + 2P(a'b') + 2P(a'b) \leq P(ab) - P(ab') + P(a'b') + P(a'b) \leq 2 \tag{28}$$

and similarly from (27):

$$-2 \leq P(ab) - P(ab') + P(a'b') + P(a'b) \leq 2 + 2P(a'b') + 2P(a'b)\,. \tag{29}$$

Hence, (28) and (29) give

$$-2 \leq S \leq 2\,,$$

where (30)

$$S = P(ab) - P(ab') + P(a'b) + P(a'b')$$

Moreover, taking (24) into account we also get

$$-2 \leq |P(ab) - P(ab')| + P(a'b) + P(a'b') \leq 2\,. \tag{31}$$

The inequality (30) can be immediately compared with quantum mechanical prediction (18). Let us choose the angles between direction $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{a}'$ and $\boldsymbol{b}'$ like in Figure 2.

From (30) we get

$$S(\theta) = 3P(\theta) - P(3\theta) \qquad \text{and} -2 \leq 3P(\theta) - P(3\theta) \leq 2\,. \tag{32}$$

But $P(\theta) = \cos 2\theta$, hence

$$-2 \leq 3\cos 2\theta - \cos 6\theta \leq 2\,. \tag{33}$$

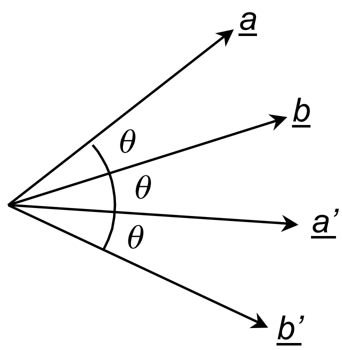


Fig. 2. Optimal fixing of vector orientations $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{a}'$, $\boldsymbol{b}'$ for spin measurements

$S(\theta)$ has its maximum for $\theta = 22.5°$ and $S(22.5°) \simeq 2.8$.

Figure 3 represents $S(\theta)$ with classical limit $-2 \leq S \leq 2$. Measurements have agreed with quantum mechanical predictions.

Coming back to the inequalities (30-31) we can say that the most exact boundaries are given by a pair of inequalities

$$|P(ab) - P(ab')| + P(a'b') + P(a'b) \leq 2$$

and (34)

$$P(ab) - P(ab') + P(a'b') + P(a'b) \geq -2$$

For the EPR paradox with electrons, the correlation function $P(ab) = -\cos\theta$. For this case we get

$$-2 \leq -3\cos\theta + \cos 3\theta \leq 2 \tag{35}$$

with $S(\theta)_{\min} = S(45°) \approx -2.8$. Similarly, in this case there is also the most visible discrepancy with classical predictions around $\theta = 45°$ (Fig. 4).

Two other Bell inequalities have been also tested experimentally. Let us assume that the photon polarisers are arranged with the same angles as in Figure 2. The probability of simultaneous registrations of two photons which passed two remote filters in A and B reads

$$w_{++} = \frac{R(ab)}{R_0}\,, \tag{36}$$

where $R(ab)$ is a number of registered pairs of photons and R_0 is the total number of registrations in four cases $w_{(++)}; w_{(--)}; w_{(+-)}; w_{(-+)}$. For

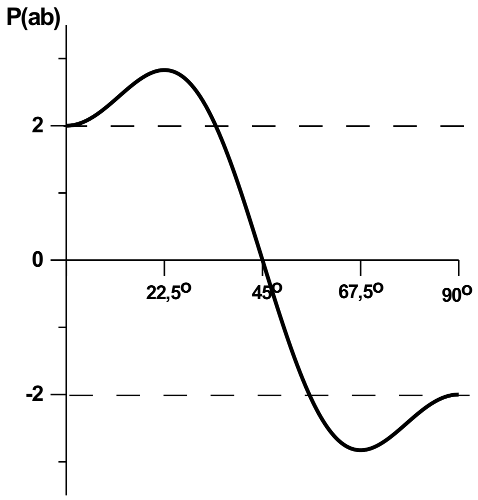


Fig. 3. The quantum function $S(\theta)$ for photons (33) with classical boundaries (dotted lines)

a hundred percentage efficiency

$$R_0 = R(0,0)\,, \tag{37}$$

where $R(0,0)$ mark the registrations without polarisers.

Let us also denote

$$R_1(a) = R(a,0) \qquad \text{and} \qquad R_2(b) = R(0,b) \tag{38}$$

that means a number of registrations with the second (first) polariser removed. Obviously we get

$$\begin{aligned} 1 &= w_{++} + w_{--} + w_{+-} + w_{-+} \\ \frac{R_1(a)}{R_0} &\equiv w_{+0} = w_{++} + w_{+-} \\ \frac{R_2(b)}{R_0} &\equiv w_{0+} = w_{++} + w_{-+}\,. \end{aligned} \tag{39}$$

Then, from (18) we get

$$\begin{aligned} P(ab) &= w_{++} + w_{--} - w_{+-} - w_{-+} = \\ &= 1 + \frac{4R(ab)}{R_0} - \frac{2R_1(a)}{R_0} - \frac{2R_2(b)}{R_0}\,. \end{aligned} \tag{40}$$

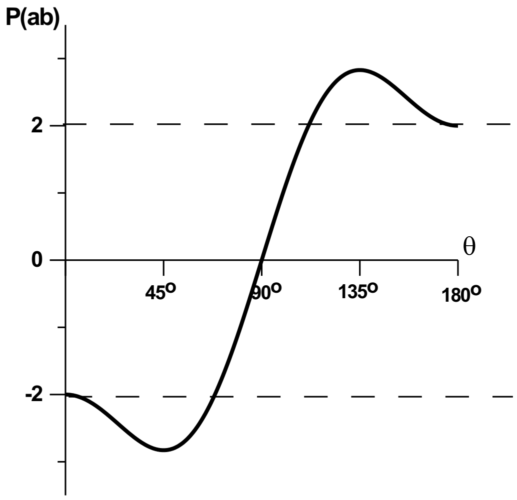


Fig. 4. The quantum function $S(\theta)$ for electrons (35) with classical boundaries (dotted lines)

and

$$\frac{R(ab)}{R_0} = \frac{1}{4}P(ab) + \frac{1}{2}\frac{R_1(a)}{R_0} + \frac{1}{2}\frac{R_2(b)}{R_0} - \frac{1}{4}\,. \tag{41}$$

For an ideal measurement

$$\frac{R_1(a)}{R_0} = \frac{R_2(b)}{R_0} = \frac{1}{2}\,. \tag{42}$$

Hence

$$\frac{R(ab)}{R_0} = 0.25P(ab) + 0.25 \tag{43}$$

or

$$P(ab) = 4\frac{R(ab)}{R_0} - 1\,. \tag{44}$$

In a practical measurement both coefficients are slightly lower, for example in [6]

$$\frac{R(ab)}{R_0} = 0.218P(ab) + 0.249\,. \tag{45}$$

From (40) and (30), after simple transformations we get

$$-1 \leq T \leq 0\,, \tag{46}$$

where

$$T = \frac{R(ab)}{R_0} - \frac{R(ab')}{R_0} + \frac{R(a'b)}{R_0} + \frac{R(a'b')}{R_0} - \frac{R_1(a')}{R_0} - \frac{R_2(b)}{R_0}\,.$$

For angles like in Figure 2 we get

$$-1 \leq \frac{3R(\theta)}{R_0} - \frac{R(3\theta)}{R_0} - \frac{R_1}{R_0} - \frac{R_2}{R_0} \leq 0\,. \tag{47}$$

This is the next generalized Bell inequality which was experimentally tested

The (47) can be further transformed. For an angle θ

$$P(\theta) = 1 + \frac{4R(\theta)}{R_0} - \frac{2R_1}{R_0} - \frac{2R_2}{R_0}$$

and with $P(\theta) = w_{++} + w_{--} - w_{+-} - w_{-+}$ we get

$$\frac{2R(\theta)}{R_0} - \frac{R_1}{R_0} - \frac{R_2}{R_0} = -(w_{+-} + w_{-+})_\theta\,.$$

The last relation can be put in (47)

$$-1 \leq \frac{R(\theta)}{R_0} - \frac{R(3\theta)}{R_0} - (w_{+-} + w_{-+})_\theta \leq 0\,. \tag{48}$$

Let us rewrite (47) for $\theta = \pi/8$ and $\theta = 3\pi/8$

$$-1 \leq \frac{3R(\pi/8)}{R_0} - \frac{R(\pi/8)}{R_0} - \frac{R_1}{R_0} - \frac{R_2}{R_0} \leq 0$$

and (49)

$$-1 \leq \frac{3R(3\pi/8)}{R_0} - \frac{R(9\pi/8)}{R_0} - \frac{R_1}{R_0} - \frac{R_2}{R_0} \leq 0\,.$$

But $\frac{R(9\pi/8)}{R_0} = \frac{R(\pi/8)}{R_0}$ and from (49) we get

$$-\frac{1}{4} \leq \frac{R(\pi/8)}{R_0} - \frac{R(3\pi/8)}{R_0} \leq \frac{1}{4}\,. \tag{50}$$

The both inequalities (47) and (50) were also obtained by Clauser et al. [5].

Now, we will demonstrate how one could get the new inequality of the similar type. Namely, from the obtained in this paper inequality (48) for $\theta = \pi/8$ and with the help of the Clauser inequality (50) we have obtained

$$\frac{1}{4} \leq (w_{+-} + w_{-+})_{\frac{\pi}{8}} \leq \frac{3}{4} \tag{51}$$

Simplicity of our inequality (51) is seen from only one angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ of both polarisers. What we need from experiments is only a number of anticoincidencies $(+-)$ and $(-+)$. We suggest here to perform proper measurements in the already constructed experimental device to check that results willl exceed the boundary limit of (51). The (51) can be also obtained for $\theta = 3\pi/8$.

5. COMPARISON WITH EXPERIMENTS

In section 3 we have given the quantum mechanics prediction for the correlation function for initially correlated photons with $S = 0$

$$P(ab) = \cos 2\theta \tag{4}$$

Then in section 4 we have gathered the family of Bell inequalities including the last one (51) obtained in this work

$$-2 \leq S \leq 2 \tag{30}$$

$$-1 \leq T \leq 0 \tag{46}$$

$$-\frac{1}{4} \leq \delta \leq \frac{1}{4} \tag{50}$$

$$\frac{1}{4} \leq X \leq \frac{3}{4} \tag{51}$$

where

$$S = P(ab) - P(ab') + P(a'b) + P(a'b')$$

$$T = \frac{R(ab)}{R_0} - \frac{R(ab')}{R_0} + \frac{R(a'b)}{R_0} + \frac{R(a'b')}{R_0} - \frac{R_1(a')}{R_0} - \frac{R_2(b)}{R_0}$$

$$\delta = \frac{R(\pi/8)}{R_0} - \frac{R(3\pi/8)}{R_0}$$

$$X = (w_{+-} + w_{-+})_{\frac{\pi}{8}}\,.$$

Angles between directions $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{a}'$, $\boldsymbol{b}'$ usually are taken as in Figure 2 with $\theta = 22.5°$ or $67.5°$. Let us gather examples of results of the three Aspect fundamental papers [6, 7, 8] with classical and quantum predictions. In the first paper [6] experimental results have been compared with the inequalities (46), (47) and (50). The R_1 and R_2 were, almost exactly taken as

$$\frac{R_1}{R_0} \approx \frac{R_2}{R_0} \approx 0.5\,.$$

and from (45) and (47) for $\theta = \pi/8$

$$T_{\mathrm{qm}} \approx 0.118\,,$$

On the other side from experimental data they obtained

$$T_{\text{exp}} = 0.126 \pm 0.0414\,.$$

Hence, the experimental value confirm the quantum mechanical results and both numbers are beyond the classical boundaries (46). Similarly, for the same angle $\theta = \pi/8$ the quantum mechanics and experimental values for δ are

$$\delta_{\text{qm}} \approx 0.25 + 5.7 \times 10^{-2} \qquad \text{and} \qquad \delta_{\text{exp}} = 0.25 + (5.8 \pm 0.2) \times 10^{-2}\,.$$

in contradiction with classical inequality (50).

In the second paper [7] the inequality (30) has been analytically analysed. The experimentally corrected function $P(\theta)$ is

$$P(\theta) = 0.955 \cos 2\theta \tag{52}$$

instead $\cos 2\theta$. Hence, for $\theta = \pi/8$

$$S_{\text{qm}}\left(\frac{\pi}{\theta}\right) = 2.70 \pm 0.05\,. \tag{53}$$

Experimental results were in high agreement with (53) and both contradicted the classical inequality (30). In the third paper [7] in which the eventual communication between two measurements in A and B has been prevented the authors obtained

$$T_{\text{exp}} = 0.101 \mp 0.020$$

which contradicts the classical inequality (46) and is in accord with quantum calculations.

6. THE CLASSICAL MODEL FOR THE EPR PARADOX

We will modify and extend the theoretical model given in the original Bell paper [3]. In the correlation function

$$P(ab) = \int d\boldsymbol{\lambda}\rho(\lambda)\, A(a\lambda)B(b\lambda) \tag{54}$$

$A(a\lambda) = \pm 1$; $B(b\lambda) = \pm 1$ and $\int d\boldsymbol{\lambda}\rho(\boldsymbol{\lambda}) = 1$.

In the general treatment nothing is assumed about a dependence of A and B on a, b, λ. Hence, the Bell inequalities already treated have the

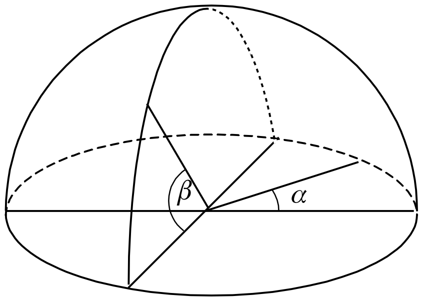


Fig. 5. Assumed λ-space as a hemisphere $|\boldsymbol{\lambda}| = 1$

general validity. But now, we assume that the λ-space is over the hemisphere $|\boldsymbol{\lambda}| = 1$, Figure 5, with the uniform probability distribution $\rho(\lambda) = \rho$. Hence

$$\int d\boldsymbol{\lambda}\rho(\lambda) = \rho \int d\boldsymbol{\lambda} = 2\pi\lambda^2\rho = 1\,, \qquad \text{or} \qquad \rho = \frac{1}{2\pi}\,. \tag{55}$$

Then

$$d\boldsymbol{\lambda} = \sin\beta d\beta d\alpha\,,$$

Let us assume that $A(a\lambda)$ and $B(b\lambda)$ do not depend on β. Hence

$$P(ab) = \rho \int\limits_0^\pi d\alpha A(a\lambda)B(b\lambda) \int\limits_0^\pi \sin\beta d\beta \qquad \text{or}$$

$$P(ab) = 2\rho \int_0^\pi d\alpha A(a\lambda)B(b\lambda) = \frac{1}{\pi}\int_0^\pi d\alpha A(a\lambda)B(b\lambda)\,. \tag{56}$$

The last integral is taken over an arc $(0, \pi)$ (Fig. 6) in such a way as to get the most similar form of $P(ab)$ for electrons ($P(ab) = -\cos\theta$) and for photons ($P(ab) = \cos 2\theta$).

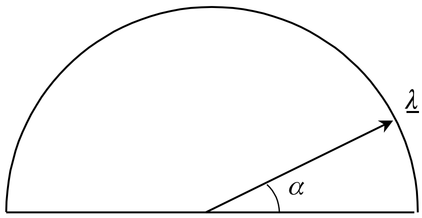


Fig. 6. The integral (56) is taken over an arc $(0, \pi)$

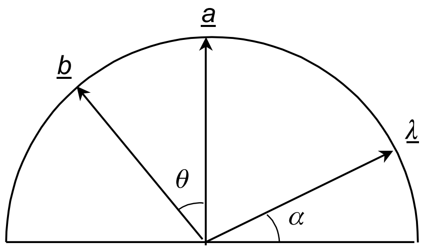


Fig. 7. The integral (56) with (57) for given $\boldsymbol{a}$ and $\boldsymbol{b}$ is equal $-1+2\theta/\pi$ (electron case)

For electrons, we assume

$$A(a\lambda) = \mathrm{sign}(\boldsymbol{a}\cdot\boldsymbol{\lambda})\,; \qquad B(b\lambda) = -\mathrm{sign}(\boldsymbol{b}\cdot\boldsymbol{\lambda})\,. \tag{57}$$

From Figure 7 we get

$$P(ab) = \frac{1}{\pi}\int_0^\pi d\alpha\, \mathrm{sign}(\boldsymbol{a}\cdot\boldsymbol{\lambda})(-\mathrm{sign}(\boldsymbol{b}\cdot\boldsymbol{\lambda})) = -1 + \frac{2\theta}{\pi}\,, \tag{58}$$

Now we can compare the classical and quantum $P(ab)$, Figure 8. We see that only for $\theta = 0°; 90°$ and $180°$

$$P_{\mathrm{class}}(ab) = P_{\mathrm{qm}}(ab)\,.$$

For a photon experiment we choose

$$A(a\lambda) = \mathrm{sign}(\boldsymbol{a}\cdot\boldsymbol{\lambda}) \qquad B(b\lambda) = \mathrm{sign}(\boldsymbol{b}'\cdot\boldsymbol{\lambda}) \qquad \text{and} \qquad \boldsymbol{b}' = R(\theta)\boldsymbol{b}\,, \tag{59}$$

where $R(\theta)$ is a rotation operator around z-axis (Fig. 9).

Now we get

$$P(ab) = \frac{1}{\pi}(\pi - 2\theta - 2\theta) = 1 - \frac{4\theta}{\pi}\,. \tag{60}$$

Hence, $P(ab)_{\mathrm{class}} = P(ab)_{\mathrm{qm}}$ for $\theta = 0°; 45°; 90°$ (Fig. 10).

Let us remark that for $\theta = \pi/8$ (the maximum difference between classical and quantum evaluation) we get from

$$P(ab) = w_{++} + w_{--} - w_{+-} - w_{-+}$$

and

$$1 = w_{++} + w_{--} - w_{+-} - w_{-+}$$

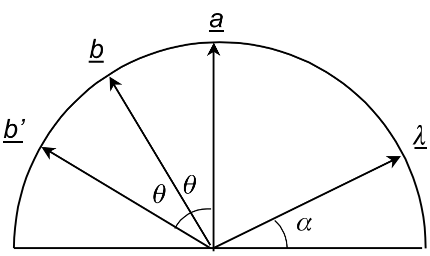


Fig. 8. Comparison of the classical correlation function $P_{\rm class}(ab) \equiv P_{\rm class}(\theta)$ with its quantum equivalent $P_{\rm qm}(\theta)$ for electrons

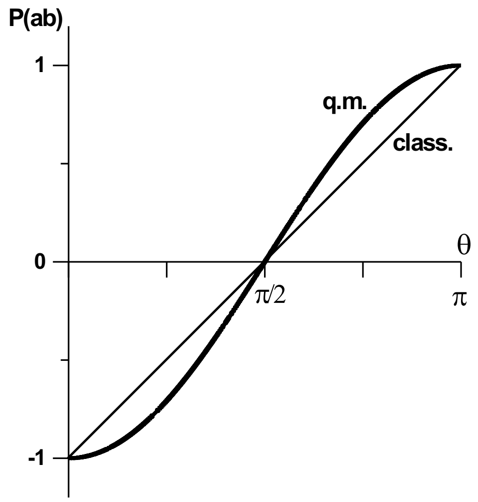


Fig. 9. The integral (56) with (59) for given $\boldsymbol{a}$ and $\boldsymbol{b}'$ is equal $1 - 4\theta/\pi$ (photon case)

the relation

$$(w_{+-} + w_{-+})_{\frac{\pi}{8}} = \frac{1}{2}\left[1 - P\left(\frac{\pi}{8}\right)\right]$$

and

$$(w_{+-} + w_{-+})_{\frac{\pi}{8}{\rm class}} = 0.25$$
$$(w_{+-} + w_{-+})_{\frac{\pi}{8}{\rm qm}} = 0.15$$

The classical value in our model reaches the lower limit of the inequality (51) while the quantum value is beyond that limit as it should be.

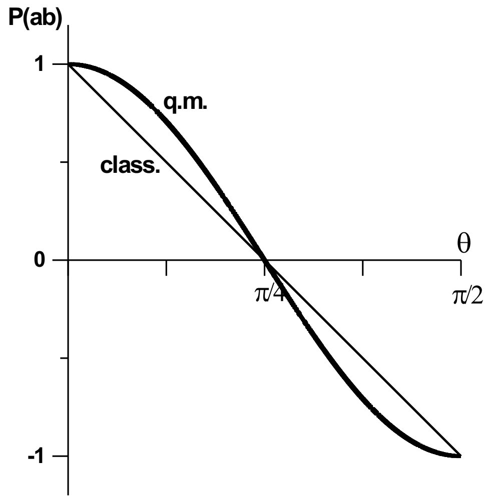


Fig. 10. Comparison of the classical correlation function $P_{\rm class}P(ab) \equiv P_{\rm class}(\theta)$ with its quantum equivalent $P_{\rm qm}(\theta)$ for photons

7. SUMMARY

In the present work:

(i) the unified treatment of the generalized Bell inequalities has been presented;

(ii) the new inequality has been constructed;

(iii) the classical model as well for the electron type EPR as for the photon type has been constructed.

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