A NEW NOTATION FOR QUANTUM MECHANICS

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In mathematical theories the question of notation, while not of primary importance, is yet worthy of careful consideration, since a good notation can be of great value in helping the development of a theory, by making it easy to write down those quantities or combinations of quantities that are important, and difficult or impossible to write down those that are unimportant. The summation convention in tensor analysis is an example, illustrating how specially appropriate a notation can be.

The notation in current use in quantum mechanics is fairly well suited to its purposes, but has some drawbacks. One has to deal with vectors in Hilbert space, representing the states of a dynamical system, and with linear operators, representing dynamical variables, and one sometimes makes calculations using the vectors and linear operators directly, treating them as abstract quantities which can be combined together algebraically according to certain rules, while at other times one works with coordinates (or *representatives*, as they are called) of these quantities. For the two styles of calculation two distinct notations are used, which do not fit together very naturally and which give rise to an awkward jump in the flow of one's thoughts when one changes from one to the other. In the present note a new notation is set up, which provides a neat and concise way of writing, in a single scheme, both the abstract quantities themselves and their coordinates, and thus leads to a unification of ideas.

A Hilbert-space vector, which was denoted in the old notation by the letter ψ, will now be denoted by a special new symbol $\rangle$. If we are concerned with a particular vector, specified by a label, a say, which would be used as a suffix to the ψ in the old notation, we write it $|a\rangle$. It may be that the label is very complicated, consisting of many letters, but we can always write down the vector conveniently in the new notation, simply by enclosing the label between $|$ on the left and $\rangle$ on the right. We have also to deal with another kind of Hilbert-space vector, the conjugate imaginary of the first kind. This was denoted in the old notation by ϕ or $\bar{\psi}$, and will now be denoted by $\langle$. If one of them is specified by a label a, we write it $\langle a|$.

A pair of vectors, one of each kind, have a symbolic product, which is a number. In the old notation such a product was denoted by $\phi\psi$, $\phi\psi_a$, $\phi_a\psi$, or $\phi_a\psi_b$, according as the vectors have labels or not. In the new notation these products will be denoted by $\langle\,\rangle$, $\langle|a\rangle$, $\langle a|\rangle$ and $\langle a|b\rangle$ respectively. Note that in the last of these it is not necessary to have the $|$ occurring twice, as a simple juxtaposition of $\langle a|$ and $|b\rangle$ would give.

Let us now introduce a representation, say the one in which each of a certain complete set of commuting observables q is diagonal. This gives us a set of basic vectors in the Hilbert space, one for each set of eigenvalues q' for the q's. These basic vectors, which were denoted in the old notation by $\psi(q')$, will now be denoted by $|q'\rangle$, the q' being treated as an ordinary label. Similarly, the conjugate imaginary basic vectors, which were denoted in the old notation by $\phi(q')$, will now be denoted by $\langle q'|$. To get the representative of any vector $|a\rangle$, we must form its product with the basic vector $\langle q'|$, which gives us $\langle q'|a\rangle$. This is similar to the bracket expression $(q'|a)$ for the representative of ψ_a in the old notation. *But while the bracket expression $(q'|a)$ of the old notation has to be introduced as an extraneous symbol, independent of what has gone before, the $\langle q'|a\rangle$ of the new notation appears naturally as a symbolic product.*

This illustrates the main feature of the new notation. With the old notation there are often two quite different ways of writing a quantity, one as a product of abstract symbols and the other by means of the bracket notation—for example, the above bracket expression $(q'|a)$ could also be written as the product $\phi(q')\,\psi_a$—but in the new notation there is always only one, and complete continuity is preserved when one passes from expressions involving abstract symbols to representatives.

This feature is illustrated also by transformation functions. If we take a second representation in which, say, each of the complete set of commuting observables ξ is diagonal, the transformation function would be written in the old notation either as the bracket expression $(q'|\xi')$ or as the symbolic product of two basic vectors $\phi(q')\,\psi(\xi')$. In the new notation these two ways of writing it coalesce into $\langle q'|\xi'\rangle$.

The development of the new notation to include linear operators and observables can be effected without difficulty. Below is a list of the various types of quantity involving a linear operator α, written on the left in the old notation and on the right in the new.

$\alpha\psi$	$\alpha\rangle$
$\alpha\psi_a$	$\alpha|a\rangle$
$\phi\alpha$	$\langle\alpha$
$\phi_a\alpha$	$\langle a|\alpha$
$\phi_a\alpha\psi$	$\langle a|\alpha\rangle$ or $\langle a|\alpha|\rangle$
$\phi\alpha\psi_a$	$\langle\alpha|a\rangle$ or $\langle|\alpha|a\rangle$
$\phi_a\alpha\psi_b$	$\langle a|\alpha|b\rangle$
$\phi(q')\,\alpha\psi(q'')$ or $(q'|\alpha|q'')$	$\langle q'|\alpha|q''\rangle$.

The last of these is the representative of a linear operator and provides a further example of a quantity that can be written in two quite different ways with the old notation, but only in one with the new. Where two forms are given for writing an expression in the new notation, the former may be used for brevity when there is no danger of α being mistaken for the label of a vector in Hilbert space, otherwise the latter must be used.

Two general rules in connexion with the new notation may be noted, namely, *any quantity in brackets* $\langle\,\rangle$ *is a number*, and *any expression containing an unclosed bracket symbol* $\langle$ *or* $\rangle$ *is a vector in Hilbert space, of the nature of a* ϕ *or* ψ *respectively*. As names for the new symbols $\langle$ and $\rangle$ to be used in speech, I suggest the words *bra* and *ket* respectively.

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